2. COMPLEX NUMBERS

Points to Remember

- Rectangular form of a complex number is $x+iy$ (or) $(x+yi)$, where x and y are real numbers.
- Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are said to be equal if and only if $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$. That is $x_1 = x_2$ and $y_1 = y_2$
- The conjugate of the complex number $x+iy$ is defined as the complex number $x-iy$.
- Properties of complex conjugates.
  1. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
  2. $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$
  3. $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$
  4. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}, z_2 \neq 0$
  5. $\text{Re}(z) = \frac{z+\overline{z}}{2}$
  6. $\text{Im}(z) = \frac{z-\overline{z}}{2i}$
  7. $\overline{(z_n)} = (\overline{z})^n$, where n is an integer
  8. z is real if and only if $z = \overline{z}$
  9. z is purely imaginary if and only if $z = -\overline{z}$
  10. $\overline{\overline{z}} = z$
- If $z = x + iy$, then $\sqrt{x^2 + y^2}$ is called modulus of z. It is denoted by $|z|$.
- Properties of Modulus of a complex number.
  1. $|z| = |\overline{z}|$
  2. $|z_1 + z_2| \leq |z_1| + |z_2|$ (Triangle inequality)
  3. $|z_1 z_2| = |z_1||z_2|$
  4. $|z_1 - z_2| \geq ||z_1| - |z_2||$
  5. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, z_2 \neq 0$
  6. $|z^n| = |z|^n$, where n is an integer
  7. $\text{Re}(z) \leq |z|$
  8. $\text{Im}(z) \leq |z|$

- Formula for finding square root of a complex number.

$$\sqrt{a+ib} = \pm\left(\sqrt{\frac{|z|+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|z|-a}{2}}\right), \text{ where } z = a+ib \text{ and } b \neq 0$$

- Let r and θ be polar coordinates of the point $P(x, y)$ that corresponds to a non-zero complex number $z = x+iy$. The polar from or trigonometric form of a complex number P is $z = r(\cos\theta + i\sin\theta)$

- **Properties of polar form**

1. If $z = r(\cos\theta + i\sin\theta)$, then $z^{-1} = \frac{1}{r}(\cos\theta - i\sin\theta)$
2. If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then
$z_1 z_2 = r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$
3. If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then $\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))$

- **De Moivre's Theorem**

1. Given any complex number $\cos\theta + i\sin\theta$ and any integer n, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
2. If x is rational, then $\cos x\theta + i\sin x\theta$ is one of the values of $(\cos\theta + i\sin\theta)^x$

- The n^th^ roots of complex number $z = r(\cos\theta + i\sin\theta)$ are

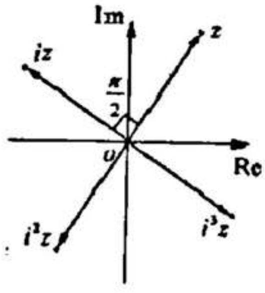
$$z^{1/n} = r^{1/n}\left(\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right), \ k = 0, 1, 2, 3, \ldots, n-1$$

Powers of imaginary unit i

$i^0 = 1, i^1 = i$	$i^2 = -1$	$i^3 = i^2 i = -i$	$i^4 = i^2 i^2 = 1$
$(i)^{-1} = \frac{1}{i} = \frac{i}{(i)^2} = -i$	$(i)^{-2} = -1$	$(i)^{-3} = i$	$(i)^{-4} = 1 = i^4$

- We note that, for any integer n, i^n has only four possible values: they correspond to values of n when divided by 4 leave the remainders 0, 1, 2, and 3.That is when the integer $n \leq -4$ or $n \geq 4$, using division algorithm, n can be written as $n = 4q + k$, $0 \leq k < 4$, k and q are integers and we write
$(i)^n = (i)^{4q+k} = (i)^{4q}(i)^k = ((i)^4)^q (i)^k = (1)^q (i)^k = (i)^k$

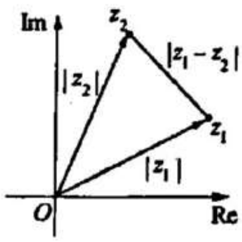
- In general, multiplication of a complex number z by i successively gives a 90° counter clockwise rotation successively about the origin.



- Complex numbers obey the laws of indices

  (i) $z^m z^n = z^{m+n}$ (ii) $\frac{z^m}{z^n} = z^{m-n}, z \neq 0$ (iii) $(z^m)^n = z^{mn}$ (iv) $(z_1 z_2)^m = z_1^m z_2^m$

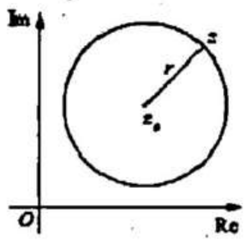
- The distance between the two points z_1 and z_2 in complex plane is $|z_1 - z_2|$



- To find the lower bound and upper bound use $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$
- $|z - z_0| = r$ is the complex form of the equation of a circle.

  (i) $|z - z_0| < r$ represents the points interior of the circle

  (ii) $|z - z_0| > r$ represents the points exterior of the circle



- If $z = 0$, the argument θ is undefined; and so it is understood that $z \neq 0$ whenever polar coordinate are used
- If the complex number $z = x + iy$ has polar coordinates (r, θ), its conjugate $\bar{z} = x - iy$ has polar coordinates $(r, -\theta)$
- Principal value of θ or principal argument of z and is denoted by Arg z.

  $-\pi < \text{Arg}(z) \leq \pi$ (or) $-\pi < \theta \leq \pi$

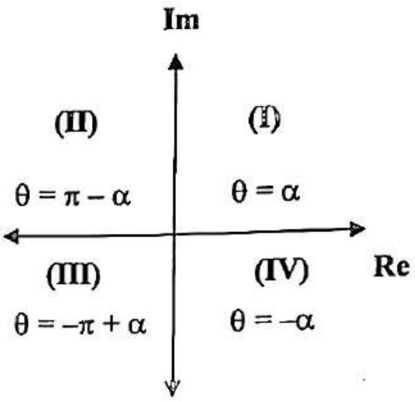
- $\arg z = \text{Arg } z + 2n\pi, n \in \mathbb{Z}$.

z	*1*	*i*	*-1*	*-i*
Arg (z)	*0*	$\frac{\pi}{2}$	π	$-\frac{\pi}{2}$
arg z	$2n\pi$	$2n\pi + \frac{\pi}{2}$	$2n\pi + \pi$	$2n\pi - \frac{\pi}{2}$

- **General rule for determining the argument θ**

Let $z = x + iy$, here $x, y \in R$

$$\alpha = \tan^{-1}\frac{|y|}{|x|}$$



- Some of the properties of arguments are

  (i) $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

  (ii) $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

  (iii) $\arg z^n = n \arg z$

  (iv) The alternate form of $\cos\theta + i\sin\theta$ is $\cos(2k\pi+\theta)+i\sin(2k\pi+\theta), k\in Z$.

- Euler's form of the complex number

  $e^{i\theta} = \cos\theta + i\sin\theta$

- $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$

  $(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$

  $(\cos\theta - i\sin\theta)^{-n} = \cos n\theta + i\sin n\theta$

  $\sin\theta + i\cos\theta = i(\cos\theta - i\sin\theta)$

- The n^{th} roots of unity $1, \omega, \omega^2, \ldots\ldots, \omega^{n-1}$ are in geometric progression with common ratio ω.
- The sum of all the n^{th} roots of unity is $1 + \omega + \omega^2 + \ldots\ldots + \omega^{n-1} = 0$
- The product of all the n^{th} roots of unity is $1.\ \omega.\ \omega^2 \ldots\ \omega^{n-1} = (-1)^{n-1}$
- All the n roots of n^{th} roots unity lie on the circumference of a circle whose centre is at the origin and radius equal to 1 and these roots divide the circle into n equal parts and form a polygon of n sides.
- In this chapter the letter ω is used for n^{th} roots of unity. Therefore the value of ω is depending on n as shown in following table.

Value of n	2	3	4	5	...	k
Value of ω	$e^{i\frac{2\pi}{2}}$	$e^{i\frac{2\pi}{3}}$	$e^{i\frac{2\pi}{4}}$	$e^{i\frac{2\pi}{5}}$	...	$e^{i\frac{2\pi}{k}}$

BOOK BACK ONE MARKS

1. $i^n + i^{n+1} + i^{n+2} + i^{n+3}$ is

1) 0 2) 1
3) −1 4) i

Solution :

$$i^n + i^{n+1} + i^{n+2} + i^{n+3}$$
$$= i^n (1 + i^1 + i^2 + i^3)$$
$$= i^n (1 + i - 1 - i)$$
$$= i^n (0)$$
$$= 0$$

2. The value of $\sum_{n=1}^{13}(i^n + i^{n+1})$ is

1) 1 + i 2) i
3) 1 4) 0

Solution :

$$= \sum_{n=1}^{13}(i^n + i^{n+1})$$
$$= \sum_{n=1}^{13} i^n\left(1+\frac{1}{i}\right) = \sum_{n=1}^{13} i^n\left(1+\frac{i}{i^2}\right)$$
$$= (1-i)\sum_{n=1}^{13} i^n$$
$$= (1-i)(i+i^2+i^3+.....+i^{13})$$
$$= (1-i)\, i = i - i^2 = 1 + i$$

3. The area of the triangle formed by the complex numbers z, iz and $z + iz$ in the Argand's diagram is

1) $\frac{1}{2}|z|^2$ 2) $|z|^2$
3) $\frac{3}{2}|z|^2$ 4) $2|z|^2$

Solution :

$$AB = |A - B| = |z - iz|$$

'D' is mid point of AB $= \dfrac{A+B}{2} = \dfrac{z+iz}{2}$

$$CD = \left|z + iz - \left(\frac{z+iz}{2}\right)\right| = \left|\frac{z+iz}{2}\right|$$

Area of the $\Delta^{le} = \frac{1}{2}$ AB x CD

$$= \frac{1}{2}|z - iz|\left|\frac{z+iz}{2}\right|$$
$$= \frac{1}{4}|z^2 - i^2 z^2|$$
$$= \frac{1}{4} \times 2\,|z|^2$$
$$= \frac{1}{2}|z|^2$$

4. The conjugate of a complex number is $\dfrac{1}{i-2}$. Then, the complex number is

1) $\dfrac{1}{i+2}$ 2) $\dfrac{-1}{i+2}$
3) $\dfrac{-1}{i-2}$ 4) $\dfrac{1}{i-2}$

Solution :

Let $\bar{z} = \dfrac{1}{i-2} \Rightarrow \bar{\bar{z}} = \overline{\left(\dfrac{1}{i-2}\right)}$

$$\bar{\bar{z}} = z = \frac{1}{-2-i} = \frac{-1}{i+2}$$

5. If $z = \dfrac{(\sqrt{3}+i)^3(3i+4)^2}{(8+6i)^2}$, then $|z|$ is equal to

1) 0 2) 1
3) 2 4) 3

Solution :

$$|z| = \frac{|\sqrt{3}+i|^3|3i+4|^2}{|8+6i|^2}$$
$$= \frac{(\sqrt{3+1})^3(\sqrt{9+16})^2}{(\sqrt{64+36})^2}$$
$$= \frac{8 \times 25}{100} = 2$$

6. If z is a non zero complex number, such that $2iz^2 = \bar{z}$ then $|z|$ is

1) $\frac{1}{2}$ 2) 1

3) 2   4) 3

Solution :

$$2iz^2 = \bar{z}$$

$$2iz\,\bar{z} = \bar{z}$$

$$2iz = 1$$

$$z = \frac{1}{2i} \times \frac{i}{i} = \frac{i}{2i^2} = -\frac{i}{2}$$

$$|z| = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

7. If $|z - 2 + i| \le 2$, then the greatest value of $|z|$ is

1) $\sqrt{3} - 2$ 2) $\sqrt{3} + 2$

3) $\sqrt{5} - 2$ 4) $\sqrt{5} + 2$

Solution :

$$|z - 2 + i| \ge |z| - |2 - i|$$

$$2 \ge |z| - \sqrt{5}$$

$$\sqrt{5} + 2 \ge |z|$$

$$|z| \le \sqrt{5} + 2$$

8. If $\left|z - \frac{3}{z}\right| = 2$, then the least value of $|z|$ is

1) 1   2) 2

3) 3   4) 5

Solution :

$$|z| = \left|z - \frac{3}{z} + \frac{3}{z}\right|$$

$$\le \left|z - \frac{3}{z}\right| + \left|\frac{3}{z}\right|$$

$$|z| - 2 - \frac{3}{|z|} \le 0$$

$$|z|^2 - 2|z| - 3 \le 0$$

$|z|$ is 3 or 1

The least value of $|z|$ is 1

9. If $|z| = 1$, then the value of $\frac{1+z}{1+\bar{z}}$ is

1) z 2) $\bar{z}$

3) $\frac{1}{z}$ 4) 1

Solution :

$$\frac{1+z}{1+\bar{z}} = \frac{1+z}{1+\frac{1}{z}} = \frac{1+z}{\frac{1+z}{z}} = z$$

$$[\because |z| = 1 \Rightarrow z\bar{z} = 1 \Rightarrow \bar{z} = \frac{1}{z}]$$

10. The solution of the equation $|z| - z = 1 + 2i$ is

1) $\frac{3}{2} - 2i$ 2) $-\frac{3}{2} + 2i$

3) $2 - \frac{3}{2}i$ 4) $2 + \frac{3}{2}i$

Solution :

Let $z = |z| - 1 - 2i$ .... (1)

$$|z|^2 = z\bar{z} = (|z| - 1 - 2i)(|z| - 1 + 2i)$$

$$= |z|^2 + |z|(-i + 2i - 1 - 2i) +$$

$$(-1 - 2i)(-1 + 2i)$$

$$|z|^2 = |z|^2 - 2|z| + 5$$

$$2|z| = 5 \Rightarrow |z| = \frac{5}{2}$$

From (1)

$$z = \frac{5}{2} - 1 - 2i$$

$$z = \frac{3}{2} - 2i$$

11. If $|z_1| = 1$, $|z_2| = 2$, $|z_3| = 3$ and $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$, then the value of $|z_1 + z_2 + z_3|$ is

1) 1   2) 2

3) 3   4) 4

Solution :

We know that

$$|z_1 + z_2 + z_3| = \frac{|9z_1z_2 + 4z_1z_3 + z_2z_3|}{|z_1||z_2||z_3|}$$

$$|z_1 + z_2 + z_3| = \frac{12}{1 \times 2 \times 3} = 2$$

12. If z is a complex number such that $z \in \mathbb{C} \setminus \mathbb{R}$ and $z + \frac{1}{z} \in R$, then $|z|$ is

1) 0 **2) 1**
3) 2 4) 3

Solution :

only when $|z| = 1$

we get $z + \frac{1}{z} \in R$

13. z_1, z_2 and z_3 are complex numbers such that $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3| = 1$ then $z_1^2 + z_2^2 + z_3^2$ is

1) 3 2) 2
3) 1 **4) 0**

Solution :

$$z_1 = \frac{1}{\bar{z}_1},\ z_2 = \frac{1}{\bar{z}_2},\ z_3 = \frac{1}{\bar{z}_3}$$

$$(z_1 + z_2 + z_3)^2$$

$$= z_1^2 + z_2^2 + z_3^2 + 2z_1z_2 + 2z_2z_3 + 2z_3z_1$$

$$0 = z_1^2 + z_2^2 + z_3^2 + 2\left(\frac{\bar{z}_3 + \bar{z}_1 + \bar{z}_2}{\bar{z}_3\ \bar{z}_1\ \bar{z}_2}\right)$$

$$0 = z_1^2 + z_2^2 + z_3^2 + 2\left(\frac{\overline{z_1 + z_2 + z_3}}{\overline{z_1\ z_2\ z_3}}\right)$$

$$0 = z_1^2 + z_2^2 + z_3^2 + 2(0)$$

$$0 = z_1^2 + z_2^2 + z_3^2$$

14. If $\frac{z-1}{z+1}$ is purely imaginary, then $|z|$ is

1) $\frac{1}{2}$ **2) 1**
3) 2 4) 3

Solution :

$\frac{z-1}{z+1}$ is purely imaginary

ie. $\text{Re}\left(\frac{z-1}{z+1}\right) = 0$

$$\text{Re}\left(\frac{x+iy-1}{x+iy+1}\right) = 0$$

$$\text{Re}\left(\frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy}\right) = 0$$

$$\left(\frac{(x-1)(x+1)+y^2}{(x+1)^2+y^2}\right) = 0$$

$$x^2 - 1 + y^2 = 0$$

$$x^2 + y^2 = 1$$

$$|z|^2 = 1$$

$$|z| = 1$$

15. If $z = x+iy$ is a complex number such that $|z+2| = |z-2|$, then the locus of z is

1) real axis **2) imaginary axis**
3) ellipse 4) circle

Solution :

Let $z = x+iy$

$$|z+2| = |z-2|$$

$$|x+iy+2| = |x+iy-2|$$

$$\sqrt{(x+2)^2+y^2} = \sqrt{(x-2)^2+y^2}$$

$$x^2+4x+4+y^2 = x^2-4x+4+y^2$$

$$8x = 0 \Rightarrow x = 0$$

i.e. imaginary axis

16. The principal argument of $\frac{3}{-1+i}$ is

1) $\frac{-5\pi}{6}$ 2) $\frac{-2\pi}{3}$
3) $\frac{-3\pi}{4}$ 4) $\frac{-\pi}{2}$

Solution :

$$\frac{3}{-1+i} = \frac{3}{-1+i} \times \frac{-1-i}{-1-i} = \frac{-3(1+i)}{2}$$

$$= \frac{1}{2}(-3-3i)$$

$\left(\frac{-3}{2}, \frac{-3}{2}\right)$ lies in III quadrant

θ is also lies in III quadrant

$\therefore\ \theta = -\pi + \alpha$

$= -\pi + \tan^{-1}\left|\frac{y}{x}\right|$

$= -\pi + \tan^{-1}\left(\frac{1}{1}\right)$

$= -\pi + \tan^{-1}(1)$

$= -\pi + \frac{\pi}{4}$

$\theta = \frac{-3\pi}{4}$

17. The principal argument of $(\sin 40° + i\cos 40°)^5$ is

1) $-110°$
2) $-70°$
3) $70°$
4) $110°$

Solution :

$(\sin 40° + i\cos 40°)^5$

$= [i(\cos 40° + i^3 \sin 40°)]^5$

$= i^5[\cos 40° - i\sin 40°]^5$

$= i[\cos 200 - i\sin 200]$

$= (\cos 90 + i\sin 90)\ (\cos 200 - i\sin 200)$

$= e^{i90} . e^{-i200} = e^{i(90-200)} = e^{-i110}$

$= \cos(-110°) + i\sin(-110°)$

18. If $(1 + i)(1 + 2i)(1 + 3i) \ldots (1 + ni) = x + iy$, then $2.5.10 \ldots (1 + n^2)$ is

1) 1
2) i
3) $x^2 + y^2$
4) $1 + n^2$

Solution :

$(1+i)(1+2i)(1+3i) \ldots (1+ni) = x + iy$,

Take modulus on both sides,

$|1+i|\,|1+2i|\,|1+3i| \ldots |1+ni| = |x+iy|$

$2.5.10 \ldots (1 + n^2) = x^2 + y^2$

19. If $\omega \neq 1$ is a cubic root of unity and $(1 + \omega)^7 = A + B\omega$, then (A, B) equals

1) (1, 0)
2) (–1, 1)
3) (0, 1)
4) (1, 1)

Solution :

$(1 + \omega)^7 = A + B\omega$

$(1 + \omega)^6 . (1 + \omega) = A + B\omega$

$(-\omega^2)^6 (1 + \omega) = A + B\omega$

$\omega^{12} (1 + \omega) = A + B\omega$

$1 + \omega = A + B\omega$

$A = 1, B = 1$

20. The principal argument of the complex number $\frac{(1+i\sqrt{3})^2}{4i(1-i\sqrt{3})}$ is

1) $\frac{2\pi}{3}$
2) $\frac{\pi}{6}$
3) $\frac{5\pi}{6}$
4) $\frac{\pi}{2}$

Solution :

$\arg\left(\frac{(1+i\sqrt{3})^2}{4i(1-i\sqrt{3})}\right)$

$= \arg(1+i\sqrt{3})^2 - \arg 4i - \arg(1-i\sqrt{3})$

$= 2\arg(1+i\sqrt{3}) - \arg(0+4i) - \arg(1-i\sqrt{3})$

$= 2\tan^{-1}\left(\frac{\sqrt{3}}{1}\right) - \tan^{-1}\left(\frac{4}{0}\right) - \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right)$

$= 2\left(\frac{\pi}{3}\right) - \frac{\pi}{2} - \left(\frac{-\pi}{3}\right) = 2\frac{\pi}{3} + \frac{\pi}{3} - \frac{\pi}{2}$

$= \pi - \frac{\pi}{2} = \frac{\pi}{2}$

21. If α and β are the roots of $x^2 + x + 1 = 0$, then $\alpha^{2020} + \beta^{2020}$ is

1) –2
2) –1
3) 1
4) 2

Solution :

$x^2 + x + 1 = 0$

$x = \frac{-1 \pm \sqrt{1-4}}{2(1)} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$

$\alpha = \frac{-1 + i\sqrt{3}}{2}, \beta = \frac{-1 - i\sqrt{3}}{2}$

$\alpha = \omega, \beta = \omega^2$

$\alpha^{2020} + \beta^{2020} = \omega^{2020} + (\omega^2)^{2020}$

$= \omega^{2020} + (\omega^{2020})^2$

$= (\omega^{2019}\omega^1) + (\omega^{2019}\omega^1)^2$

$= \omega + \omega^2 = -1$

$= -1$

22. The product of all four values of

$$\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{\frac{3}{4}} \text{ is}$$

1) –2   2) –1

3) 1   4) 2

Solution :

$$\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{\frac{3}{4}}=\text{cis}\left(2k\pi+\frac{\pi}{3}\right)^{\frac{3}{4}}$$

$$=\text{cis}\left(\frac{6k\pi+\pi}{3}\right)^{\frac{3}{4}}$$

$$=\text{cis}\left(\frac{6k\pi+\pi}{4}\right), k=0,1,2,3$$

$$k=0\Rightarrow \text{cis}\frac{\pi}{4};\ k=2\Rightarrow \text{cis}\frac{13\pi}{4}$$

$$k=1\Rightarrow \text{cis}\frac{7\pi}{4};\ k=3\Rightarrow \text{cis}\frac{19\pi}{4}$$

$$\text{product}=\text{cis}\left(\frac{\pi}{4}+\frac{7\pi}{4}+\frac{13\pi}{4}+\frac{19\pi}{4}\right)$$

$$=\text{cis}\left(\frac{40\pi}{4}\right)=\text{cis}\,10\pi$$

$$=\cos 10\pi+i\ \sin 10\pi=1+0i$$

$$=1$$

23. If $\omega \neq 1$ is a cubic root of unity and $\begin{vmatrix}1 & 1 & 1\\ 1 & -\omega^2-1 & \omega^2\\ 1 & \omega^2 & \omega^7\end{vmatrix}=3k$, then k is equal to

1) 1   2) –1

3) $\sqrt{3}\,i$ 4) $-\sqrt{3}\,i$

Solution :

$$\begin{vmatrix}1 & 1 & 1\\ 1 & -\omega^2-1 & \omega^2\\ 1 & \omega^2 & \omega^7\end{vmatrix}=3k$$

$$\begin{vmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{vmatrix}=3k$$

$$\Rightarrow\left(\omega^2-\omega^4\right)-1\left(\omega-\omega^2\right)+1\left(\omega^2-\omega\right)=3k$$

$$\omega^2-\omega-\omega+\omega^2+\omega^2-\omega=3k$$

$$3\omega^2-3\omega=3k\Rightarrow 3(\omega^2-\omega)=3k$$

$$k=\omega^2-\omega$$

$$=\frac{1}{2}\left[-1-i\sqrt{3}+1-i\sqrt{3}\right]$$

$$=\frac{1}{2}\left[-2i\sqrt{3}\right]$$

$$k=-i\sqrt{3}$$

$$=-\sqrt{3}i$$

24. The value of $\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10}$ is

1) $\text{cis}\frac{2\pi}{3}$ 2) $\text{cis}\frac{4\pi}{3}$

3) $-\text{cis}\frac{2\pi}{3}$ 4) $-\text{cis}\frac{4\pi}{3}$

Solution :

Let $1+\sqrt{3}i=r\,(\cos\theta+i\sin\theta)$

$$r=\sqrt{1+3}=2$$

$\left(1,\sqrt{3}\right)$ lies in I quadrant, θ' is also lies in I quadrant

$\therefore\ \theta=\alpha$

$$\theta=\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)\Rightarrow\theta=\frac{\pi}{3}$$

$$1+\sqrt{3}i=2\,\text{cis}\frac{\pi}{3}=2\,e^{i\frac{\pi}{3}}$$

Replace is 'i' by '–i'

$$1-\sqrt{3}i=2\,\text{cis}\left(\frac{-\pi}{3}\right)=2\,e^{-i\frac{\pi}{3}}$$

$$\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10}=\left(\frac{2e^{i\frac{\pi}{3}}}{2e^{-i\frac{\pi}{3}}}\right)^{10}$$

$$=\left(e^{i\frac{\pi}{3}}.e^{i\frac{\pi}{3}}\right)^{10}$$

$$=\left(e^{i2\frac{\pi}{3}}\right)^{10}=e^{i20\frac{\pi}{3}}$$

$$=\text{cis}\ 20\frac{\pi}{3}$$

$= \text{cis}\left(6\pi + \frac{2\pi}{3}\right)$

$= \text{cis}\ \frac{2\pi}{3}$

25. If $\omega = \text{cis}\frac{2\pi}{3}$, then the number of distinct root of $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$

1) 1 2) 2

3) 3 4) 4

Solution :

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$

$[(z+1)((z+\omega^2)(z+\omega)-1)-1]$

$-\omega(z\omega+\omega^2-\omega^2)+\omega^2(\omega-z\omega^2-\omega^4)=0$

$\Rightarrow (z+1)(z^2+z\omega+z\omega^2+\omega^2)-z\omega^2+$

$\omega^3-z\omega^4-\omega^6=0$

$\Rightarrow z^2+z^2\omega+z^2\omega^2+z^2+z\omega+z\omega^2$

$-z\omega^2+1-z\omega-1=0$

$\Rightarrow z^2(1+\omega+\omega^2+z)=0$

$\Rightarrow z^3=0$

$\Rightarrow z=0, 0, 0$

The number of distinct roots is 1

BOOK SUMS ***(Exercise and Examples) :***

1. Simplify the following

(i) i^7 (ii) i^{1729} (iii) $i^{-1924}+i^{2018}$ (iv) $\sum_{n=1}^{102} i^n$ (v) $i\ i^2\ i^3 \ldots\ i^{40}$

2. Simplify the following

(i) $i^{1947}+i^{1950}$ (ii) $i^{1948}-i^{-1869}$ (iii) $\sum_{n=1}^{12} i^n$

(iv) $i^{59}+\frac{1}{i^{59}}$ (v) $i\ i^2\ i^3 \ldots\ i^{2000}$ (vi) $\sum_{n=1}^{10} i^{n+50}$

3. Find the value of the real numbers x and y, if the complex numbers $(2+i)x+(1-i)y+2i-3$ and $x+(-1+2i)y+1+i$ are equal.

4. Evaluate the following if $z = 5-2i$ and $w=-1+3i$

(i) z + w (ii) z − i w (iii) 2z + 3w

(iv) z w (v) $z^2+2zw+w^2$ (vi) $(z+w)^2$

5. Given the complex number $z = 2+3i$, represent the complex numbers in Argand diagram.

(i) z, iz and z + iz (ii) z, $-iz$ and z − iz

6. Find the values of the real numbers x and y, if the complex numbers $(3-i)x-(2-i)y+2i+5$ and $2x+(-1+2i)y+3+2i$ are equal.

7. If $z_1 = 1-3i$, $z_2=-4i$ and $z_3=5$, show that

(i) $(z_1+z_2)+z_3 = z_1+(z_2+z_3)$ (ii) $(z_1z_2)z_3 = z_1(z_2z_3)$

3. THEORY OF EQUATIONS

Points to Remember

- For the quadratic equation $ax^2+bx+c=0$,
  i) $\Delta=b^2-4ac>0$ iff the roots are real and distinct.
  ii) $\Delta=b^2-4ac<0$ iff the equation has no real roots.
  iii) $\Delta=b^2-4ac=0$ iff the roots are real and equal .

Fundamental theorem of algebra :

- Every polynomial equation of degree n has at least one root in C.

Complex conjugate root theorem

- If a complex number z_0 is a root of a polynomial equation with real co-efficients, then complex conjugate $\overline{z_0}$ is also a root.
- If $p+\sqrt{q}$ is a root of a quadratic equation then $p-\sqrt{q}$ is also a root of the same equation where p, q are rational and $\sqrt{q}$ is irrational.
- If $\sqrt{p}+\sqrt{q}$ is a root of a polynomial equation then $\sqrt{p}-\sqrt{q}$, $-\sqrt{p}+\sqrt{q}$ and $-\sqrt{p}-\sqrt{q}$ are also roots of the same equation.
- If the sum of the co-efficients in $P(x)=0$ is P(1). Then 1 is a root of $P(x)=0$.
- If the sum of the co-efficients of odd powers = sum of the co-efficients of even powers, then -1 is a root of $P(x)=0$.

Rational root theorem

- Let $a_n x^n+\dots\dots+a_1x+a_0$ with $a_n\neq 0$, $a_0\neq 0$ be a polynomial with integer co-efficients. If $\frac{p}{q}$ with $(p,q)=1$ is a root of the polynomial, then p is a factor of a_0 and q is a factor of a_n

Reciprocal polynomial

- A polynomial $P(x)$ of degree n is said to be a reciprocal polynomial if one of the conditions is true
  (i) $P(x)=x^nP\left(\frac{1}{x}\right)$ (ii) $P(x)=-x^nP\left(\frac{1}{x}\right)$
- A change of sign in the co-efficients is said to occur at the j^{th} power of x in $P(x)$ if the co-efficient of x^{j+1} and the co-efficient of x^j (or) co-efficient of x^{j-1} and co-efficient of x^j are of different signs.
- **Vieta's formula for quadratic equation :**
  If α, β are the roots of $ax^2+bx+c=0$ then $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$.

  Also, x^2-x(sum of the roots) + product of the roots = 0

- **Vieta's formula for polynomial of degree 3.**

  Co-efficient of $x^2 = -(\alpha+\beta+\gamma)$ where α, β, γ are its roots

  Co-efficient of $x = \alpha\beta+\beta\gamma+\gamma\alpha$ and constant term $= -\alpha\beta\gamma$

  Also, $x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma=0$

- **Vieta's formula for polynomial equation of degree n > 3.**

  Co-efficient of $x^{n-1} = \Sigma_1 = -\Sigma\alpha_1$

  Co-efficient of $x^{n-2} = \Sigma_2 = \Sigma\alpha_1\,\alpha_2$

  Co-efficient of $x^{n-3} = \Sigma_3 = -\Sigma\alpha_1\,\alpha_2\,\alpha_3$

  Co-efficient of $x = \Sigma_{n-1} = (-1)^{n-1}\,\Sigma\alpha_1\,\alpha_2 \ldots\ldots \alpha_{n-1}$

  Co-efficient of x^0 = constant term $= \Sigma_n = (-1)^n\,\alpha_1\,\alpha_2 \ldots\ldots \alpha_n$

  A polynomial equation $a_nx^n+a_{n-1}x^{n-1}+\ldots\ldots+a_1x+a_0=0$ $(a_n\neq 0)$ is a reciprocal equation iff one of the following statements is true.

  i) $a_n=a_0, a_{n-1}=a_1, a_{n-2}=a_2, \ldots\ldots$

  ii) $a_n=-a_0, a_{n-1}=-a_1, a_{n-2}=-a_2, \ldots\ldots$

Descartes rule

- If p is the number of positive zeros of a polynomial $P(x)$ with real co-efficients and s is the number of sign changes in Co-efficient of $P(x)$, then $s-p$ is a non negative even integer.

Bounds for the number of real and imaginary roots

- Let m denote the number of sign changes in coefficients of $P(x)$ of degree n and $P(x)$ has atmost m positive zeros.
- Let k denote the number of sign changes in coefficients of $P(-x)$ of degree n and $P(x)$ has atmost k negative zeros.
- Then $P(x)$ has atleast (m+k) real roots and atleast n–(m+k) imaginary roots.

BOOK BACK ONE MARKS

1. A zero of $x^3 + 64$ is

1) 0 2) 4
3) $4i$ **4) -4**

Solution :

$$x^3 + 64 = 0$$
$$x^3 = -64$$
$$x^3 = (-4)^3$$
$$x = -4$$

2. If f and g are polynomials of degrees m and n respectively, and if $h(x) = (f \circ g)(x)$, then the degree of h is

1) mn 2) $m + n$
3) m^n 4) n^m

Solution :

Let $f(x) = x^m$, $g(x) = x^n$

$$h(x) = (fog)(x) = f(g(x)) = f(x^n) = (x^n)^m = x^{mn}$$

The degree of $h(x)$ is mn.

3. A polynomial equation in x of degree n always has

1) n distinct roots 2) n real roots
3) n imaginary roots 4) at most one root.

Solution :

Every polynomial equation of degree n has at least one root in C.

$\therefore$ n imaginary roots

4. If α, β and γ are the roots of $x^3 + px^2 + qx + r$, then $\sum \frac{1}{\alpha}$ is

1) $-\frac{q}{r}$ 2) $-\frac{p}{r}$
3) $\frac{q}{r}$ 4) $-\frac{q}{p}$

Solution :

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = \frac{q}{1} = q$$
$$\alpha\beta\gamma = \frac{-d}{a} = \frac{-r}{1} = -r$$
$$\sum \frac{1}{\alpha} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$$
$$= \frac{\beta\gamma + \beta\gamma + \alpha\beta}{\alpha\beta\gamma} = \frac{q}{-r}$$
$$= -\frac{q}{r}$$

(Book answer is wrong)

5. According to the rational root theorem, which number is not possible rational root of $4x^7 + 2x^4 - 10x^3 - 5$?

1) -1 2) $\frac{5}{4}$
3) $\frac{4}{5}$ 4) 5

Solution :

The given polynomial equation is $4x^7 + 2x^4 - 10x^2 - 5 = 0$

$a_n = 4$, $a_0 = -5$

If $\frac{p}{q}$ is a root of the polynomial, then as $(p,q) = 1$, p must divide -5 and q must divide 4.

The possible values of p are ±1, ±5 and the possible values of q are ±1, ±2, ±4.

Using these p and q, we can form only fractions $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{5}{1}, \pm\frac{5}{2}, \pm\frac{5}{4}$

Hence $\frac{4}{5}$ is not a possible root of the equation

6. The polynomial $x^3 - kx^2 + 9x$ has three real zeros if and only if, k satisfies

1) $|k| \le 6$ 2) $k = 0$

3) $|k| > 6$ 4) $|k| \ge 6$

Solution :

$P(x) = x^3 - kx^2 + 9x$

$= x(x^2 - kx + 9)$

Clearly, $P(x)$ has one root as zero which real. The other roots are determined by the factor $x^2 - kx + 9$

This factor will give real roots if $b^2 - 4ac \ge 0$

For real roots $b^2 - 4ac \ge 0$

$(-k)^2 - 4(1)(9) \ge 0$

$\Rightarrow k^2 \ge 36$

$\Rightarrow |k| \ge 6$

7. The number of real numbers in $[0, 2\pi]$ satisfying $\sin^4 x - 2\sin^2 x + 1$ is

1) 2 2) 4

3) 1 4) ∞

Solution :

The given equation is

$\Rightarrow \sin^4 x - 2\sin^2 x + 1 = 0$

$\Rightarrow (\sin^2 x - 1)^2 = 0$

$\Rightarrow \sin^2 x - 1 = 0$

$\Rightarrow \sin^2 x = 1$

$\Rightarrow \frac{1 - \cos 2x}{2} = 1$

$\Rightarrow 1 - \cos 2x = 2$

$\Rightarrow -\cos 2x = 2 - 1 = 1$

$\Rightarrow \cos 2x = -1$

$\Rightarrow 2x = (2n+1)\pi, n \in z$

$\Rightarrow x = (2n+1)\frac{\pi}{2}$

$x = \frac{\pi}{2}, \frac{3\pi}{2} \in [0, 2\pi]$

Hence two real numbers are there in the interval $[0, 2\pi]$ satisfying the given equation.

8. If $x^3 + 12x^2 + 10ax + 1999$ definitely has a positive zero, if and only if

1) $a \ge 0$ 2) $a > 0$

3) $a < 0$ 4) $a \le 0$

Solution :

The equation $x^3 + 12x^2 + 10ax + 1999$ has a positive root if it has at least one change of sign. So, a must be negative.

$\therefore a < 0$

9. The polynomial $x^3 + 2x + 3$ has

1) one negative and two imaginary zeros

2) one positive and two imaginary zeros

3) three real zeros

4) no zeros

Solution :

$p(x) = x^3 + 2x + 3$

$p(x)$ has no sign change

$p(-x) = (-x)^3 + 2(-x) + 3$

$= -x^3 - 2x + 3$

$p(-x)$ has only one sign change and atmost one negative root.

$p(x)$ has no positive root and at most one negative root. Degree of $p(x)$ is 3

$\Rightarrow$ imaginary roots = 3 – 1 = 2,

The polynomial has one negative and two imaginary roots.

10. The number of positive zeros of the polynomial $\sum_{j=0}^{n} {}^nC_r(-1)^r x^r$ is

1) 0 2) n

3) $< n$ 4) r

Solution :

$\sum_{j=0}^{n} {}^nC_r(-1)^r x^r = {}^nC_0(-1)^0 x^0 + {}^nC_1(-1)^1 x^1$

$+ {}^nC_2(-1)^2 x^2 + \ldots + {}^nC_n(-1)^n x^n$

$= 1 - nx^1 + {}^nC_2 x^2 - {}^nC_3 x^3 + \ldots + x^n$

Since its degree is n and it has n changes of sign, the number of positive roots are n.

4. INVERSE TRIGONOMETRIC FUNCTIONS

Points to Remember

- **Inverse Trigonometric Functions**

Inverse sine function	*Inverse cosine function*	*Inverse tangent function*	*Inverse cosecant function*	*Inverse secant function*	*Inverse cot function*
Domain $[-1, 1]$	Domain $[-1, 1]$	Domain R	Domain $(-\infty, -1] \cup [1, \infty)$	Domain $(-\infty, -1] \cup [1, \infty)$	Domain R
Range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	Range $[0, \pi]$	Range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	Range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$	Range $[0, \pi] - \left\{\frac{\pi}{2}\right\}$	Range $(0, \pi)$
not a periodic function	not a periodic function	not a periodic function	not a periodic function	not a periodic function	not a periodic function
odd function	neither even nor odd function	odd function	odd function	neither even nor odd function	neither even nor odd function
strictly increasing function	strictly decreasing function	strictly increasing function	strictly decreasing function with respect to its domain.	strictly decreasing function with respect to its domain.	strictly decreasing function
one to one function	one to one function	one to one function	one to one function	one to one function	one to one function

- **Properties of inverse Trigonometric Functions.**

Property I

(i) $\sin^{-1}(\sin\theta) = \theta$, if $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $\cos^{-1}(\cos\theta) = \theta$, if $\theta \in [0, \pi]$

(iii) $\tan^{-1}(\tan\theta) = \theta$, if $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(iv) $\text{cosec}^{-1}(\text{cosec}\theta) = \theta$, if $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$

(v) $\sec^{-1}(\sec\theta) = \theta$, if $\theta \in [0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$

(vi) $\cot^{-1}(\cot\theta) = \theta$, if $\theta \in (0, \pi)$

Property II

(i) $\sin(\sin^{-1}x) = x$, if $x \in [-1, 1]$

(ii) $\cos(\cos^{-1}x) = x$, if $x \in [-1, 1]$

(iii) $\tan(\tan^{-1}x) = x$, if $x \in R$

(iv) $\text{cosec}(\text{cosec}^{-1}x) = x$, if $\in R \setminus (1,1)$

(v) $\sec(\sec^{-1}x) = x$, if $x \in R \setminus (-1,1)$

(vi) $\cot(\cot^{-1}x) = x$, if $x \in R$

Property III (Reciprocal inverse identities)

(i) $\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}\, x$, if $x \in R \setminus (-1,1)$

(ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec x$, if $x \in R \setminus (-1,1)$

(iii) $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x & \text{if } x > 0 \\ -\pi + \cot^{-1} x & \text{if } x < 0 \end{cases}$

Property-IV(Reflection identities)

(i) $\sin^{-1}(-x) = -\sin^{-1} x$, if $x \in [-1, 1]$

(ii) $\tan^{-1}(-x) = -\tan^{-1} x$, if $x \in R$.

(iii) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x$, if $|x| \geq 1$ or $x \in R \setminus (-1, 1)$

(iv) $\cos^{-1}(-x) = \pi - \cos^{-1} x$, if $x \in [-1, 1]$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1} x$, if $|x| \geq 1$ or $x \in R \setminus (-1, 1)$

(vi) $\cot^{-1}(-x) = \pi - \cot^{-1} x$, if $x \in R$.

Property-V (cofunction inverse identities)

(i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, $x \in [-1, 1]$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$, $x \in R$

(iii) $\text{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$, $x \in R \setminus (-1, 1)$ or $|x| \geq 1$

Property-VI

(i) $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$ where either $x^2 + y^2 \leq 1$ or $xy < 0$.

(ii) $\sin^{-1} x - \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$ where either $x^2 + y^2 \leq 1$ or $xy > 0$.

(iii) $\cos^{-1} x + \cos^{-1} y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$, if $x + y \geq 0$.

(iv) $\cos^{-1} x - \cos^{-1} y = \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)$, if $x \leq y$

(v) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, if $xy < 1$.

(vi) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$, if $xy > -1$.

Property-VII

(i) $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right), |x| < 1$

(ii) $2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), x \geq 0$

(iii) $2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right), |x| \leq 1$

Property-VIII

(i) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\sin^{-1}x$, if $|x| \leq \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$.

(ii) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\cos^{-1}x$, if $\frac{1}{\sqrt{2}} \leq x \leq 1$.

Property-IX

(i) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2}$, if $0 \leq x \leq 1$.

(ii) $\sin^{-1}x = -\cos^{-1}\sqrt{1-x^2}$, if $-1 \leq x < 0$.

(iii) $\sin^{-1}x = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$, if $-1 < x < 1$.

(iv) $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2}$, if $0 \leq x \leq 1$.

(v) $\cos^{-1}x = \pi - \sin^{-1}\sqrt{1-x^2}$, if $-1 \leq x < 0$.

(vi) $\tan^{-1}x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$, if $x > 0$.

Property-X

(i) $3\sin^{-1}x = \sin^{-1}\left(3x-4x^3\right), x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

(ii) $3\cos^{-1}x = \cos^{-1}\left(4x^3-3x\right), x \in \left[\frac{1}{2}, 1\right]$

6. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$, the value of $x^{2017} + y^{2018} + z^{2019} - \frac{9}{x^{101} + y^{101} + z^{101}}$ is

1) 0 2) 1

3) 2 4) 3

Solution :

Given

$$\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$$

w.k.t $\sin^{-1}x$ attains its maximum value at $x = 1$ and its maximum value is $\frac{\pi}{2}$.

Hence the above equation has only one solution $x = y = z = 1$.

Hence,

$$x^{2017} + x^{2018} + x^{2019} - \frac{9}{x^{101} + y^{101} + z^{101}}$$

$$= 1^{2017} + 1^{2018} + 1^{2019} - \frac{9}{1^{101} + 1^{101} + 1^{101}}$$

$$= 1 + 1 + 1 - \frac{9}{1+1+1}$$

$$= 3 - \frac{9}{3}$$

$$= 3 - 3 = 0$$

7. If $\cot^{-1}x = \frac{2\pi}{5}$ for some $x \in R$, the value of $\tan^{-1}x$ is

1) $-\frac{\pi}{10}$ 2) $\frac{\pi}{5}$

3) $\frac{\pi}{10}$ 4) $-\frac{\pi}{5}$

Solution :

$$\cot^{-1}x = \frac{2\pi}{5} \quad \left(\therefore \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right)$$

$$\frac{\pi}{2} - \tan^{-1}x = \frac{2\pi}{5}$$

$$\frac{\pi}{2} - \frac{2\pi}{5} = \tan^{-1}x$$

$$\tan^{-1}x = \frac{5\pi - 4\pi}{10} = \frac{\pi}{10}$$

8. The domain of the function defined by $f(x) = \sin^{-1}\sqrt{x-1}$ is

1) [1, 2] 2) [–1, 1]

3) [0, 1] 4) [–1, 0]

Solution :

Given $f(x) = \sin^{-1}\sqrt{x-1}$

First of all the expression $\sqrt{x-1}$ must be real and it is real only when

$$x - 1 \geq 0 \Rightarrow 0 \leq x - 1 \text{ ------- (1)}$$

The domain of $\sin^{-1}x$ is $-1 \leq x \leq 1$

(i.e) $-1 \leq \sqrt{x-1} \leq 1$

hence $\sqrt{x-1} \leq 1$ (or) $x - 1 \leq 1$ -------- (2)

Combining both the inequalities (1) and (2) we get,

$$0 \leq x - 1 \leq 1 \Rightarrow 1 \leq x \leq 2$$

9. If $x = \frac{1}{5}$, the value of $\cos(\cos^{-1}x + 2\sin^{-1}x)$ is

1) $-\sqrt{\frac{24}{25}}$ 2) $\sqrt{\frac{24}{25}}$

3) $\frac{1}{5}$ **4) $-\frac{1}{5}$**

Solution :

Given $x = \frac{1}{5}$

$$\cos\left(\cos^{-1}x + 2\sin^{-1}x\right)$$

$$= \cos\left(\cos^{-1}x + \sin^{-1}x + \sin^{-1}x\right)$$

$$= \cos\left(\frac{\pi}{2} + \sin^{-1}x\right)$$

$$\left(\because \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta\right)$$

$$= -\sin\left(\sin^{-1}x\right)$$

$$= -x$$

$$= \frac{-1}{5}$$

10. $\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{2}{9}\right)$ is equal to

1) $\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$ 2) $\frac{1}{2}\sin^{-1}\left(\frac{3}{5}\right)$

3) $\frac{1}{2}\tan^{-1}\left(\frac{3}{5}\right)$ 4) $\tan^{-1}\left(\frac{1}{2}\right)$

Solution :

w.k.t $\tan^{-1}A+\tan^{-1}B=\tan^{-1}\left(\frac{A+B}{1-AB}\right)$

$\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{2}{9}\right)$

$=\tan^{-1}\left[\frac{\frac{1}{4}+\frac{2}{9}}{1-\left(\frac{1}{4}\times\frac{2}{9}\right)}\right]$

$=\tan^{-1}\left[\frac{\frac{9+8}{36}}{\frac{36-2}{36}}\right]$

$=\tan^{-1}\left[\frac{17}{34}\right]=\tan^{-1}\left[\frac{1}{2}\right]$

11. If the function $f(x)=\sin^{-1}(x^2-3)$, then x belongs to

1) $[-1,1]$

2) $[\sqrt{2},\ 2]$

3) $[-2,-\sqrt{2}]\cup[\sqrt{2},2]$

4) $[-2,-\sqrt{2}]\cap[\sqrt{2},2]$

Solution :

$f(x)=\sin^{-1}(x^2-3)$ then $-1\le x^2-3\le 1$

$\Rightarrow -1+3\le x^2\le 1+3$

$\Rightarrow 2\le x^2\le 4$

$\Rightarrow \pm\sqrt{2}\le x\le\pm 2\Rightarrow\sqrt{2}\le x\le 2$ and

$\Rightarrow -2\le x\le-\sqrt{2}$

Hence $x\in[-2,-\sqrt{2}]\cup[\sqrt{2},2]$

12. If $\cot^{-1}2$ and $\cot^{-1}3$ are two angles of a triangle, then the third angle is

1) $\frac{\pi}{4}$ 2) $\frac{3\pi}{4}$

3) $\frac{\pi}{6}$ 4) $\frac{\pi}{3}$

Solution :

$\cot^{-1}2+\cot^{-1}3=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}$

$=\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)}\right)$

$=\tan^{-1}\left(\frac{\frac{3+2}{6}}{\frac{6-1}{6}}\right)$

$=\tan^{-1}\left(\frac{5}{5}\right)=\tan^{-1}(1)=\frac{\pi}{4}$

The sum of two angles $\frac{\pi}{4}$

Sum of three angles in triangle $=\pi$

Hence the third angle $=\pi-\frac{\pi}{4}$

$=\frac{3\pi}{4}$

13. $\sin^{-1}\left(\tan\frac{\pi}{4}\right)-\sin^{-1}\left(\sqrt{\frac{3}{x}}\right)=\frac{\pi}{6}$. Then x is a root of the equation

1) $x^2-x-6=0$ 2) $x^2-x-12=0$

3) $x^2+x-12=0$ 4) $x^2+x-6=0$

Solution :

$\sin^{-1}\left(\tan\frac{\pi}{4}\right)-\sin^{-1}\left(\sqrt{\frac{3}{x}}\right)=\frac{\pi}{6}$.

$\sin^{-1}(1)-\sin^{-1}\left(\sqrt{\frac{3}{x}}\right)=\frac{\pi}{6}$

$\frac{\pi}{2}-\sin^{-1}\left(\sqrt{\frac{3}{x}}\right)=\frac{\pi}{6}$

$$\frac{\pi}{2}-\frac{\pi}{6}=\sin^{-1}\left(\sqrt{\frac{3}{x}}\right)$$

$$\sin^{-1}\left(\sqrt{\frac{3}{x}}\right)=\frac{\pi}{3}$$

$$\sqrt{\frac{3}{x}}=\sin\frac{\pi}{3}$$

$$\sqrt{\frac{3}{x}}=\frac{\sqrt{3}}{2}$$

squaring on both sides $\frac{3}{x}=\frac{3}{4}$

$x=4$

Moreover, $x=4$ is a root of the equation

$x^2-x-12=0$

14. $\sin^{-1}\left(2\cos^2 x-1\right)+\cos^{-1}\left(1-2\sin^2 x\right)=$

1) $\frac{\pi}{2}$ 2) $\frac{\pi}{3}$

3) $\frac{\pi}{4}$ 4) $\frac{\pi}{6}$

Solution :

$\sin^{-1}\left(2\cos^2 x-1\right)+\cos^{-1}\left(1-2\sin^2 x\right)$

$=\sin^{-1}(\cos 2x)+\cos^{-1}(\cos 2x)=\frac{\pi}{2}$

[since $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$]

15. If $\cot^{-1}\left(\sqrt{\sin\alpha}\right)+\tan^{-1}\left(\sqrt{\sin\alpha}\right)=u$, then cos $2u$ is equal to

1) $\tan^2\alpha$ 2) 0

3) –1 4) $\tan 2\alpha$

Solution :

w.k.t $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$

Given

$\cot^{-1}\left(\sqrt{\sin\alpha}\right)+\tan^{-1}\left(\sqrt{\sin\alpha}\right)=u$

$u=\frac{\pi}{2}$

$\cos 2u=\cos\left(2\times\frac{\pi}{2}\right)$

$=\cos\pi$

$=-1$

16. If $|x|\le 1$, then $2\tan^{-1}x-\sin^{-1}\frac{2x}{1+x^2}$ is equal to

1) $\tan^{-1}x$ 2) $\sin^{-1}x$

3) 0 4) π

Solution :

W.K.T, $2\tan^{-1}x=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$\therefore 2\tan^{-1}x-\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$=2\tan^{-1}x-2\tan^{-1}x$

$=0$

17. The equation $\tan^{-1}x-\cot^{-1}x=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ has

1) no solution

2) unique solution

3) two solutions

4) infinite number of solutions

Solution :

$\tan^{-1}x-\cot^{-1}x=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\tan^{-1}x-\left(\frac{\pi}{2}-\tan^{-1}x\right)=\frac{\pi}{6}$

$\tan^{-1}x-\frac{\pi}{2}+\tan^{-1}x=\frac{\pi}{6}$

$2\tan^{-1}x=\frac{\pi}{6}+\frac{\pi}{2}$

$=\frac{2\pi}{3}$

$\tan^{-1}x=\frac{\pi}{3}$

$x=\tan\frac{\pi}{3}$

$x=\sqrt{3}$

Hence the given equation has only one solution (or) unique solution.

18. If $\sin^{-1} x + \cot^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2}$, then x is equal to

1) $\frac{1}{2}$ 2) $\frac{1}{\sqrt{5}}$

3) $\frac{2}{\sqrt{5}}$ 4) $\frac{\sqrt{3}}{2}$

Solution :

Let $\cot^{-1}\left(\frac{1}{2}\right) = \alpha$ .... (1)

$\cot \alpha = \frac{1}{2}$

$\tan \alpha = 2$

$\sec^2 \alpha = 1 + \tan^2 \alpha = 1 + 2^2 = 5$

$\sec \alpha = \sqrt{5}$

$\cos \alpha = \frac{1}{\sqrt{5}}$

$\alpha = \cos^{-1} \frac{1}{\sqrt{5}}$ .... (2)

From (1) and (2), we get

$\cot^{-1} \frac{1}{2} = \cos^{-1} \frac{1}{\sqrt{5}}$

Given

$\sin^{-1} x + \cot^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2}$

$\sin^{-1} x + \cos^{-1} \frac{1}{\sqrt{5}} = \frac{\pi}{2}$

$x = \frac{1}{\sqrt{5}}$

19. If $\sin^{-1} \frac{x}{5} + \operatorname{cosec}^{-1} \frac{5}{4} = \frac{\pi}{2}$, then the value of x is

1) 4 2) 5

3) 2 **4) 3**

Solution :

$\sin^{-1} \frac{x}{5} + \operatorname{cosec}^{-1} \frac{5}{4} = \frac{\pi}{2}$

$\sin^{-1} \frac{x}{5} + \sin^{-1} \frac{4}{5} = \frac{\pi}{2}$

$\sin^{-1} \frac{x}{5} = \frac{\pi}{2} - \sin^{-1} \frac{4}{5}$

$\frac{x}{5} = \sin\left(\frac{\pi}{2} - \sin^{-1} \frac{4}{5}\right)$

$= \cos\left(\sin^{-1} \frac{4}{5}\right)$

$= \cos\left(\cos^{-1} \frac{3}{5}\right)$

(since $\sin \theta = \frac{4}{5} = \frac{opp}{hyp}$

$\cos \theta = \frac{adj}{hyp} = \frac{\sqrt{5^2 + 4^2}}{5} = \frac{3}{5}$)

$\Rightarrow \frac{x}{5} = \frac{3}{5}$

$\Rightarrow x = 3$

20. $\sin(\tan^{-1} x)$, $|x| < 1$ is equal to

1) $\frac{x}{\sqrt{1-x^2}}$ 2) $\frac{1}{\sqrt{1-x^2}}$

3) $\frac{1}{\sqrt{1+x^2}}$ **4)** $\frac{x}{\sqrt{1+x^2}}$

Solution :

Let $\tan^{-1} x = \theta$

$\tan \theta = x$

$1 + \tan^2 \theta = 1 + x^2$

$\sec^2 \theta = 1 + x^2$

$\sec \theta = \sqrt{1+x^2}$

$\cos \theta = \frac{1}{\sqrt{1+x^2}}$

Now, $\tan \theta = \frac{\sin \theta}{\cos \theta}$

$\sin \theta = \tan \theta \times \cos \theta$

$= x \cdot \frac{1}{\sqrt{1+x^2}}$

$\sin\left(\tan^{-1} x\right) = \frac{x}{\sqrt{1+x^2}}$

5. TWO DIMENSIONAL ANALYTICAL GEOMETRY-II

Points to Remembers :

- Equation of the circle in a standard form is $(x-h)^2+(y-k)^2=r^2$
  (i) centre (h,k) (ii) radius ' r '
- Equation of a circle in general form is $x^2+y^2+2gx+2fy+c=0$
  (i) centre (–g, –f) (ii) radius = $\sqrt{g^2+f^2-c}$
- The circle through the intersection of the line $\ell x+my+n=0$ and the circle $x^2+y^2+2gx+2fy+c=0$ is $x^2+y^2+2gx+2fy+c+\lambda(\ell x+my+n)=0,\ \lambda\in R^1$
- Equation of a circle with (x_1,y_1) and (x_2,y_2) as extremities of one of the diameters is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$
- Equation of tangent at (x_1,y_1) on circle $x^2+y^2+2gx+2fy+c=0$ is
  $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$
- Equation of normal at (x_1,y_1) on circle $x^2+y^2+2gx+2fy+c=0$ is
  $yx_1-xy_1+g(y-y_1)-f(x-x_1)=0$

Table - 1

Tangent and normal

Curve	*Equation*	*Equation of tangent*	*Equation of normal*
circle	$x^2+y^2=a^2$	i) **Cartesian form** $xx_1+yy_1=a^2$ ii) **Parametric form** $x\cos\theta+y\sin\theta=a$	i) **Cartesian form** $xy_1-yx_1=0$ ii) **Parametric form** $x\sin\theta-y\cos\theta=0$
Parabola	$y^2=4ax$	i) **Cartesian form** $yy_1=2a(x+x_1)$ ii) **Parametric form** $yt=x+at^2$	i) **Cartesian form** $xy_1+2y=2ay_1+x_1y_1$ ii) **Parametric form** $y+xt=at^3+2at$
Ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	i) **Cartesian form** $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ ii) **Parametric form** $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$	i) **Cartesian form** $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$ ii) **Parametric form** $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$
Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	i) **Cartesian form** $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$ ii) **Parametric form** $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$	i) **Cartesian form** $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$ ii) **Parametric form** $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$

Table – 2

Condition for the sine $y = mx + c$ to be a tangent to the conics

Curve	*Equation*	*Condition to be tangent*	*Point of contact*	*Equation of tangent*
circle	$x^2 + y^2 = a^2$	$c^2 = a^2(1+m^2)$	$\left(\frac{\mp am}{\sqrt{1+m^2}}, \frac{\pm a}{\sqrt{1+m^2}}\right)$	$y = mx \pm a\sqrt{1+m^2}$
Parabola	$y^2 = 4ax$	$c = \frac{a}{m}$	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$	$y = mx + \frac{a}{m}$
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$c^2 = a^2m^2 + b^2$	$\left(\frac{-a^2m}{c}, \frac{b^2}{c}\right)$	$y = mx \pm \sqrt{a^2m^2 + b^2}$
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$c^2 = a^2m^2 - b^2$	$\left(\frac{-a^2m}{c}, \frac{-b^2}{c}\right)$	$y = mx \pm \sqrt{a^2m^2 - b^2}$

Table – 3

Parametric forms

Conic	*Parametric equations*	*Parameter*	*Range of parameter*	*Any point on the conic*
Circle	$x = a\cos\theta$ $y = a\sin\theta$	θ	$0 \le \theta \le 2\pi$	'θ' or $(a\cos\theta, a\sin\theta)$
Parabola	$x = at^2$ $y = 2at$	t	$-\infty < t < \infty$	't' or $(at^2, 2at)$
Ellipse	$x = a\cos\theta$ $y = b\sin\theta$	θ	$0 \le \theta \le 2\pi$	'θ' or $(a\cos\theta, b\sin\theta)$
Hyperbola	$x = a\sec\theta$ $y = b\tan\theta$	θ	$-\pi \le \theta \le \pi$ except $\theta = \pm\frac{\pi}{2}$	'θ' or $(a\sec\theta, b\tan\theta)$

Identifying the conic from the general equation of conic $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

The graph of the second degree equation is one of a circle, parabola, an ellipse, a hyperbola, a point, an empty set, a single line or a pair of lines. When,

- $A = C = 1$, $B = 0$, $D = -2h$, $E = -2k$, $F = h^2 + k^2 - r^2$ the general equation reduces to $(x-h)^2 + (y-k)^2 = r^2$, which is a circle.
- $B = 0$ and either A or $C = 0$, the general equation yields a parabola under study, at this level.
- $A \ne C$ and A and C are of the same sign the general equation yields an ellipse.
- $A \ne C$ and A and C are of opposite signs the general equation yields a hyperbola.
- $A = C$ and $B = D = E = F = 0$, the general equation yields a point $x^2 + y^2 = 0$.
- $A = C = F$ and $B = D = E = 0$, the general equation yields an empty set $x^2 + y^2 + 1 = 0$, as there is no real solution.
- $A \ne 0$ or $C \ne 0$ and others are zeros, the general equation yield coordinate axes.
- $A = -C$ and rests are zero, the general equation yields a pair of lines $x^2 - y^2 = 0$.

BOOK BACK ONE MARKS

1. The equation of the circle passing through (1, 5) and (4, 1) and touching y-axis is $x^2+y^2-5x-6y+9+\lambda(4x+3y-19)=0$ where λ is equal to

(1) $0, -\frac{40}{9}$ (2) 0

(3) $\frac{40}{9}$ (4) $\frac{-40}{9}$

Solution :

$$x^2+y^2-5x+4\lambda x+3\lambda y-6y-19\lambda+9=0$$

$$x^2+y^2+2x\left(2\lambda-\frac{5}{2}\right)+2y\left(\frac{3\lambda}{2}-3\right)+9-19\lambda=0$$

centre – y axis distance = radius

$$2\lambda-\frac{5}{2}=\sqrt{\left(2\lambda-\frac{5}{2}\right)^2+\left(\frac{3\lambda}{2}-3\right)^2+19\lambda-9}$$

$$\left(2\lambda-\frac{5}{2}\right)^2=\left(2\lambda-\frac{5}{2}\right)^2+$$

$$\left(\frac{3\lambda}{2}-3\right)^2+19\lambda-9$$

$$0=\frac{9\lambda^2}{4}+9-9\lambda+19\lambda-9$$

$$\frac{9\lambda^2}{4}+10\lambda=0$$

$$\lambda\left(\frac{9\lambda+40}{4}\right)=0$$

$\lambda=0$

$9\lambda=-40$

$$\lambda=\frac{-40}{9}$$

2. The eccentricity of the hyperbola whose latus rectum is 8 and conjugate axis is equal to half the distance between the foci is

(1) $\frac{4}{3}$ (2) $\frac{4}{\sqrt{3}}$

(3) $\frac{2}{\sqrt{3}}$ (4) $\frac{3}{2}$

Solution :

$$\frac{2b^2}{a}=8$$

$$b^2=\frac{8a}{2}$$

$b^2=4a$

$$2b=\frac{1}{2}(2ae)$$

$2b=ae$

$a^2e^2=4b^2$

$=4(4a)$ $(\because b^2=4a)$

$=16a$

$$e^2=\frac{16a}{a^2}=\frac{16}{a}$$

$b^2=a^2e^2-a^2$

$4a=16a-a^2$

$a^2-16a+4a=0$

$a^2-12a=0$

$a(a-12)=0$

$a=12$

$$e^2=\frac{16}{12}=\frac{4}{3}$$

$$e=\frac{2}{\sqrt{3}}$$

3. The circle $x^2+y^2=4x+8y+5$ intersects the line $3x-4y=m$ at two distinct points if

(1) $15<m<65$ (2) $35<m<85$

(3) $-85<m<-35$ **(4) $-35<m<15$**

Solution :

$$x^2+y^2-4x-8y-5=0$$

centre $=(-g,-f)$

$=(2,4)$

radius $=\sqrt{4+16+5}$

$=5$

$$\text{distance}=\left|\frac{3x_1-4y_1-m}{\sqrt{(3)^2+(-4)^2}}\right|<5$$

$$\left|\frac{3(2)-4(4)-m}{\sqrt{9+16}}\right| < 5$$

$|6-16-m| < 25$

$|10+m| < 25$

$-25 < 10+m < 25$

$-35 < m < 15$

4. The length of the diameter of the circle which touches the x -axis at the point (1, 0) and passes through the point (2, 3).

(1) $\frac{6}{5}$ (2) $\frac{5}{3}$

(3) $\mathbf{\frac{10}{3}}$ (4) $\frac{3}{5}$

Solution : $P(x, y)$

centre $(1, a)$

radius = $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$a = \sqrt{(x-1)^2+(y-a)^2}$

If passes (2, 3)

$a^2 = (2-1)^2+(3-a)^2$

$a^2 = 1+9-6a+a^2$

$6a = 10$

$a = \frac{10}{6}$

$a = \frac{5}{3}$

Diameter = 2a = $\frac{10}{3}$

5. The radius of the circle $3x^2+by^2+4bx-6by+b^2=0$ is

(1) 1 (2) 3

(3) $\mathbf{\sqrt{10}}$ (4) $\sqrt{11}$

Solution :

Co-efficient of x^2 = co-efficient of y^2.

b = 3

$3x^2+3y^2+12x-18y+9=0$

$x^2+y^2+4x-6y+3=0$

$g=2,\ f=-3,\ c=3$

$r = \sqrt{g^2+f^2-c}$

$= \sqrt{4+9-3}$

$r = \sqrt{10}$

6. The centre of the circle inscribed in a square formed by the lines $x^2-8x-12=0$ and $y^2-14y+45=0$ is

(1) (4, 7) (2) (7, 4)

(3) (9, 4) (4) (4,9)

Solution :

$x^2-8x-12=0$

$x^2-8x+16-16-12=0$

$(x-4)^2 = 28$

$x-4 = \pm 2\sqrt{7}$

$x = 4\pm 2\sqrt{7}$

$y^2-14y+45 = 0$

$(y-9)(y-5) = 0$

$y = 5, 9$

$(4-2\sqrt{7}, 5)$ $(4+2\sqrt{7}, 9)$

centre $= \left(\frac{4+2\sqrt{7}+4-2\sqrt{7}}{2}, \frac{5+9}{2}\right)$

$= (4, 7)$

7. The equation of the normal to the circle $x^2+y^2-2x-2y+1=0$ which is parallel to the line $2x+4y=3$ is

(1) $\mathbf{x+2y=3}$ (2) $x+2y+3=0$

(3) $2x+4y+3=0$ (4) $x-2y+3=0$

Solutions :

$x^2+y^2-2x-2y+1=0$

centre = $(-g, -f)$ = (1, 1)

normal equation

$2x+4y=k$

It passes (1, 1)

$2+4 = k$

$k = 6$

$2x+4y = 6$

$x+2y = 3$

8. If $P(x,y)$ be any point on $16x^2+25y^2=400$ with foci F_1 (3, 0) and F_2 (–3, 0) then $PF_1 + PF_2$ is

(1) 8 (2) 6

(3) 10 (4) 12

Solution :

$16x^2+25y^2=400$

$\frac{x^2}{25}+\frac{y^2}{16}=1$

$a^2 = 25$

$a = 5$

$F_1P + F_2P = 2a = 2(5) = 10$

9. The radius of the circle passing through the point (6 , 2) two of whose diameters are $x+y=6$ and $x+2y=4$ is

(1) 10 (2) $2\sqrt{5}$

(3) 6 (4) 4

Solution :

$x+y = 6$ $\quad x-2=6$

$x+2y = 4$ $\quad x=8$

$-y = 2$ $\quad$ centre (8, –2)

$y = -2$

radius $=\sqrt{(6-8)^2+(2+2)^2}$

$=\sqrt{(-2)^2+(4)^2}$

$=\sqrt{4+16}$

$=\sqrt{20}$

$=2\sqrt{5}$

10. The area of quadrilateral formed with foci of the hyperbolas $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ is

(1) $4(a^2+b^2)$ (2) $2(a^2+b^2)$

(3) a^2+b^2 (4) $\frac{1}{2}(a^2+b^2)$

Solution :

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

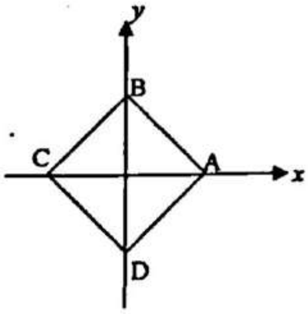
A(ae, 0), C(–ae, 0)

$e=\sqrt{\frac{a^2+b^2}{a^2}}$

$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$

B(0, be), D(0, –be)

$e=\sqrt{\frac{a^2+b^2}{b^2}}$



Area of quadrilateral ABCD

$=\frac{1}{2}d_1 \times d_2$

$=\frac{1}{2}2ae \times 2be$

$=\frac{a\sqrt{a^2+b^2}}{a} \times \frac{2b\sqrt{a^2+b^2}}{b}=2(a^2+b^2)$

11. If the normals of the parabola $y^2=4x$ drawn at the end points of its latus rectum are tangents to the circle $(x-3)^2+(y+2)^2=r^2$, then the value of r^2 is

(1) 2 (2) 3

(3) 1 (4) 4

Solution :

$y^2 = 4x$

$y^2 = 4ax$

$a = 1$

Length of latus rectum 4a = 4

End Points are (1, 2), (1, –2)

($\because$ (a, 2a), (a, –2a))

Tangent equation at(1, 2)

$yy_1 = 4\left(\frac{x+x_1}{2}\right)$

$yy_1 = 2(x+x_1)$

$y = (x+1)$

$x-y+1 = 0$

Normal equation

$x+y+k = 0$

It passes (1, 2)

$1 + 2 + k = 0$

$k = -3$

Normal equation $x+y-3=0$, centre : (3, –2)

radius $=\left|\frac{3-2-3}{\sqrt{1^2+1^2}}\right|=\frac{2}{\sqrt{2}}=\sqrt{2}$

$r=\sqrt{2}$ $\quad r^2=2$

12. If $x+y=k$ is a normal to the parabola $y^2=12x$, then the value of k is

(1) 3 (2) -1

(3) 1 **(4) 9**

Solution :

$y^2 = 12x$

a $= 3$

y $= -x+k$

m $= -1$

c $= k$

condition for the normal equation of $y^2 = 4ax$

$k = -2am - am^3$

$k = -2(3)(-1) - 3(-1)^3$

$k = 6+3$

$k = 9$

13. The ellipse $E_1 : \frac{x^2}{9}+\frac{y^2}{4}=1$ is inscribed in a rectangle R whose sides are parallel to the coordinate axes. Another ellipse E_2 passing through the point (0, 4) circumscribes the rectangle R. The eccentricity of the ellipse is

(1) $\frac{\sqrt{2}}{2}$ (2) $\frac{\sqrt{3}}{2}$

(3) $\frac{1}{2}$ (4) $\frac{3}{4}$

Solution :

Equation of ellipse

$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$

It passes (0, 4)

$\frac{(0)^2}{b^2}+\frac{(4)^2}{a^2}=1$

$\frac{16}{a^2} = 1$

$a^2 = 16$

$\frac{x^2}{b^2}+\frac{y^2}{16} = 1$

It passes (3, 2)

$\frac{9}{b^2}+\frac{4}{16} = 1$

$\frac{9}{b^2} = 1-\frac{1}{4}$

$\frac{9}{b^2} = \frac{3}{4}$

$b^2 = \frac{36}{3}$

$b^2 = 12$

$e = \sqrt{\frac{a^2-b^2}{a^2}}$

$e = \sqrt{\frac{4}{16}} = \frac{1}{2}$

14. Tangents are drawn to the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$ parallel to the straight line $2x-y=1$. One of the points of contact of tangents on the hyperbola is

(1) $\left(\frac{9}{2\sqrt{2}},\frac{-1}{\sqrt{2}}\right)$ (2) $\left(\frac{-9}{2\sqrt{2}},\frac{1}{\sqrt{2}}\right)$

(3) $\left(\frac{9}{2\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ (4) $\left(3\sqrt{3},-2\sqrt{2}\right)$

Solution :

$a^2 = 9,\quad b^2 = 4 \qquad m = 2$

$c = \pm\sqrt{a^2m^2-b^2}$

$= \pm\sqrt{9\times4-4} = \pm\sqrt{32}$

Take $c = -4\sqrt{2}$

Point of contact $= \left(\frac{-a^2m}{c},\frac{-b^2}{c}\right)$

$= \left(\frac{-9\times2}{-4\sqrt{2}},\frac{-4}{-4\sqrt{2}}\right)$

$= \left(\frac{9}{2\sqrt{2}},\frac{1}{\sqrt{2}}\right)$

15. The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ having centre at (0, 3) is

(1) $x^2+y^2-6y-7=0$

(2) $x^2+y^2-6y+7=0$

(3) $x^2+y^2-6y-5=0$

(4) $x^2+y^2-6y+5=0$

Solutions :

$a^2 = 16, \qquad b^2 = 9$

$e = \sqrt{\frac{a^2-b^2}{a^2}} = \sqrt{\frac{16-9}{16}} = \frac{\sqrt{7}}{4}$

Foci $(\pm ae, 0) = (\pm\sqrt{7}, 0)$

equation of the circle with centre (0, 3) is

$(x-0)^2+(y-3)^2=r^2$ ……….. (1)

It passes $(\sqrt{7},0)$

$(\sqrt{7})^2+3^2=r^2$

$r^2=16$

$(1)\Rightarrow x^2+(y-3)^2=16$

$x^2+y^2-6y-7=0$

16. Let C be the circle with centre at (1, 1)and radius = 1. If T is the circle centered at $(0, y)$ passing through the origin and touching the circle C externally, then the radius of T is equal to

(1) $\frac{\sqrt{3}}{\sqrt{2}}$ (2) $\frac{\sqrt{3}}{2}$

(3) $\frac{1}{2}$ (4) $\frac{1}{4}$

Solution :

$r_1=1$ let $r_2=y$

$c_1c_2=r_1+r_2$

$c_1\ (1, 1)$

$c_2\ (0, y)$

$\sqrt{(1-0)^2+(1-y)^2}=1+y$

$1+1^2+y^2-2y=(1+y)^2$

$1+1+y^2-2y=1+y^2+2y$

$1=2y+2y$

$4y=1$

$y=\frac{1}{4}$

17. Consider an ellipse whose centre is of the origin and its major axis is along x-axis. If its eccentricity is $\frac{3}{5}$ and the distance between its foci is 6, then the area of the quadrilateral inscribed in the ellipse with diagonals as major and minor axis of the ellipse is

(1) 8 (2) 32

(3) 80 (4) 40

Solution :

$2ae=6$

$ae=3$

$e=\frac{3}{5}$

$a\left(\frac{3}{5}\right)=3$

$a=5$

$b^2=a^2-a^2e^2=25-9$

$b^2=16$

$b=4$

Area of the quadrilateral $=\frac{1}{2}d_1\times d_2$

$=\frac{1}{2}(2a)(2b)$

$=5\times 2\times 4$

$=40$ sq.units

18. Area of the greatest rectangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

(1) 2*ab* (2) *ab*

(3) $\sqrt{ab}$ (4) $\frac{a}{b}$

Solution :

$x=a\cos\theta$

$y=b\sin\theta$

Area $=\ell\times b$

$=2a\cos\theta\times 2b\sin\theta$

$A=2ab\sin 2\theta$

$A'=4ab\cos 2\theta$

$A'=0$

$\cos 2\theta=0$

$\theta=45°$

$A=2ab\sin 2(45)$

$A=2ab$

19. An ellipse has OB as semi minor axes, F and F′ its foci and the angle FBF′ is a right angle. Then the eccentricity of the ellipse is

(1) $\frac{1}{\sqrt{2}}$ (2) $\frac{1}{2}$

(3) $\frac{1}{4}$ (4) $\frac{1}{\sqrt{3}}$

Solution :

$FF'^2=BF^2+BF'^2$

$(2ae)^2=\left(\sqrt{(ae-0)^2+(0-b)^2}\right)^2+\left(\sqrt{(-ae-0)^2+(0-b)^2}\right)^2$

$4a^2e^2=a^2e^2+b^2+a^2e^2+b^2$

$4a^2e^2-2a^2e^2=2b^2$

$2b^2=2a^2e^2$

$b^2=a^2e^2$

$$a^2 - a^2e^2 = a^2e^2$$
$$a^2 = 2a^2e^2$$
$$e^2 = \frac{1}{2}$$
$$e = \frac{1}{\sqrt{2}}$$

20. The eccentricity of the ellipse $(x-3)^2 + (y-4)^2 = \frac{y^2}{9}$ is

(1) $\frac{\sqrt{3}}{2}$ (2) $\frac{1}{3}$

(3) $\frac{1}{3\sqrt{2}}$ (4) $\frac{1}{\sqrt{3}}$

Solution :

$$FP^2 = e^2PM^2$$

$$(x-3)^2 + (y-4)^2 = \left(\frac{1}{3}\right)^2 \left(\frac{0+y+0}{\sqrt{0+1}}\right)^2$$

$$= \left(\frac{1}{3}\right)^2 \left(\frac{\ell x + my + n}{\sqrt{\ell^2 + m^2}}\right)^2$$

$$e = \frac{1}{3}$$

21. If the two tangents drawn from a point P to the parabola $y^2 = 4x$ are at right angles then the locus of P is

(1) $2x + 1 = 0$ (2) $x = -1$

(3) $2x - 1 = 0$ (4) $x = 1$

Solution :

$y^2 = 4x$

length of latus rectum

$4a = 4$

$a = 1$

equation of directrix $x = -a$

$x = -1$

22. The circle passing through (1,–2) and touching the axis of x at (3, 0) passing through the point

(1) (–5, 2) (2) (2,–5)

(3) (5,–2) (4) (–2, 5)

Solution :

$$(x-3)^2 + (y-k)^2 = k^2$$

It passes (1, –2)

$$(1-3)^2 + (-2-k)^2 = k^2$$
$$4 + 4 + 4k + k^2 = k^2$$
$$4k = -8$$
$$k = -2$$
$$(x-3)^2 + (y+2)^2 = 4$$

lies on (5, –2)

$$2^2 + 0^2 = 4$$
$$4 = 4$$

23. The locus of a point whose distance from (–2,0) is $\frac{2}{3}$ times its distance from the line $x = \frac{-9}{2}$ is

(1) a parabola (2) a hyperbola

(3) an ellipse (4) a circle

Solution :

$$FP = \frac{2}{3} PM$$

$$\frac{FP}{PM} = \frac{2}{3} < 1$$

$e < 1$ it is an ellipse

24. The values of m for which the line $y = mx + 2\sqrt{5}$ touches the hyperbola $16x^2 - 9y^2 = 144$ are the roots of $x^2 - (a+b)x - 4 = 0$, then the value of $(a + b)$ is

(1) 2 (2) 4

(3) 0 (4) -2

Solution :

$$\frac{x^2}{9} - \frac{y^2}{16} = 1 \qquad y = mx + 2\sqrt{5}$$

$a^2 = 9$, $b^2 = 16$ $c = 2\sqrt{5}$

condition $c^2 = a^2m^2 - b^2$

$$(2\sqrt{5})^2 = 9m^2 - 16$$
$$20 = 9m^2 - 16$$
$$9m^2 = 36$$
$$m^2 = 4$$
$$m = \pm 2$$

Take the values of m roots

$a = 2$ $b = -2$

product = –4

$a + b = 2 - 2 = 0$

25. If the coordinates at one end of a diameter of the circle $x^2+y^2-8x-4y+c=0$ are (11, 2), the coordinates of the other end are

(1) (–5, 2) (2) (2, –5)

(3) (5, –2) (4) (–2, 5)

Solution :

$$\left(\frac{11+x}{2}, \frac{2+y}{2}\right) = (4, 2)$$

$$\frac{11+x}{2} = 4$$

$$x = 8 - 11$$

$$x = -3$$

$$\frac{2+y}{2} = 2$$

$$2 + y = 4$$

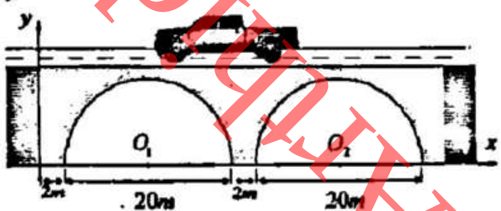
$$y = 2$$

(Book answer is wrong)

Correct answer is (–3, 2)

BOOK SUMS ***(Exercise and Examples) :***

1. Find the general equation of a circle with centre (–3, –4) and radius 3 units.
2. Find the equation of the circle described on the chord $3x+y+5=0$ of the circle $x^2+y^2=16$ as diameter.
3. Determine whether $x+y-1=0$ is the equation of a diameter of the circle $x^2+y^2-6x+4y+c=0$ for all possible values of c.
4. Find the general equation of the circle whose diameter is the line segment joining the points (–4, –2) and (1, 1).
5. Examine the position of the point (2, 3) with respect to the circle $x^2+y^2-6x-8y+12=0$.
6. The line $3x+4y-12=0$ meets the coordinate axes at A and B. Find the equation of the circle drawn on AB as diameter.
7. A line $3x+4y+10=0$ cuts a chord of length 6 units on a circle with centre of the circle (2, 1). Find the equation of the circle in general form.
8. A circle of radius 3 units touches both the axes. Find the equations of all possible circles formed in the general form.
9. Find the centre and radius of the circle $3x^2+(a+1)y^2+6x-9y+a+4=0$.
10. Find the equation of the circle passing through the points (1, 1), (2, –1) and (3, 2).
11. Find the equations of the tangent and normal to the circle $x^2+y^2=25$ at $P(-3, 4)$.
12. If $y = 4x + c$ is a tangent to the circle $x^2+y^2=9$, find c.
13. A road bridge over an irrigation canal have two semi circular vents each with a span of $20m$ and the supporting pillars of width $2m$. Use figure to write the equations that represent the semi-verticular vents.



14. Obtain the equation of the circles with radius 5 cm and touching x-axis at the origin in general form.
15. Find the equation of the circle with centre (2, –1)and passing through the point (3, 6) in standard form.
16. Find the equation of circles that touch both the axes and pass through (–4, –2) in general form.
17. Find the equation of the circle with centre (2, 3) and passing through the intersection of the lines $3x-2y-1=0$ and $4x+y-27=0$
18. Obtain the equation of the circle for which (3, 4) and (2, –7)are the ends of a diameter.

6. APPLICATIONS OF VECTOR ALGEBRA

Points to remember :

- For a given set of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, the scalar $(\vec{a}\times\vec{b}).\vec{c}$ is called a scalar triple product of $\vec{a}$, $\vec{b}$, $\vec{c}$.
- The volume of the parallelopiped formed by using the three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, as co-terminus edges is given by $|(\vec{a}\times\vec{b}).\vec{c}|$.
- The scalar triple product of three non-zero vectors is zero if and only if the three vectors are coplanar.
- Three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar, if and only if there exist scalars r, s, t ∈ R such that atleast one of them is non-zero and $r\vec{a}+s\vec{b}+t\vec{c}=\vec{0}$
- If $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{p}$, $\vec{q}$, $\vec{r}$ are any two systems of three vectors, and if $\vec{p}=x_1\vec{a}+y_1\vec{b}+z_1\vec{c}$, $\vec{q}=x_2\vec{a}+y_2\vec{b}+z_2\vec{c}$ and $\vec{r}=x_3\vec{a}+y_3\vec{b}+z_3\vec{c}$ then $[\vec{p},\vec{q},\vec{r}]=\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}[\vec{a},\vec{b},\vec{c}]$
- For a given set of three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ the vector $\vec{a}\times(\vec{b}\times\vec{c})$ is called vector triple product.
- For any three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ we have $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$.
- Parametric form of the vector equation of a straight line that passes through a given point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r}=\vec{a}+t\vec{b}$, where t ∈ R.
- Cartesian equations of a straight line that passes through the point (x_1, y_1, z_1) and parallel to a vector with direction ratios b_1, b_2, b_3 are $\frac{x-x_1}{b_1}=\frac{y-y_1}{b_2}=\frac{z-z_1}{b_3}$.
- Any point on the line $\frac{x-x_1}{b_1}=\frac{y-y_1}{b_2}=\frac{z-z_1}{b_3}$ is of the form $(x_1+tb_1, y_1+tb_2, z_1+tb_3)$, t ∈ R.
- Parametric form of vector equation of a straight line that passes through two given points with position vectors $\vec{a}$ and $\vec{b}$ is $\vec{r}=\vec{a}+t(\vec{b}-\vec{a})$, t ∈ R.
- Cartesian equations of a line that passes through two given points (x_1, y_1, z_1) and (x_2, y_2, z_2) are $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$.
- If θ is the acute angle between two straight lines $\vec{r}=\vec{a}+s\vec{b}$ and $\vec{r}=\vec{c}+t\vec{d}$, then $\theta=\cos^{-1}\left(\frac{|\vec{b}.\vec{d}|}{|\vec{b}||\vec{d}|}\right)$
- Two lines are said to be coplanar if they lie in the same plane.
- Two lines in space are called skew lines if they are not parallel and do not intersect
- The shortest distance between the two skew lines is the length of the line segment perpendicular to both the skew lines.
- The shortest distance between the two skew lines $\vec{r}=\vec{a}+s\vec{b}$ and $\vec{r}=\vec{c}+t\vec{d}$ is

$$\delta = \frac{|(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})|}{|\vec{b}\times\vec{d}|}, \text{ where } |\vec{b}\times\vec{d}| \neq 0.$$

- Two straight lines $\vec{r}=\vec{a}+s\vec{b}$ and $\vec{r}=\vec{c}+t\vec{d}$ intersect each other if $(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})=0$
- The shortest distance between the two parallel lines $\vec{r}=\vec{a}+s\vec{b}$ and $\vec{r}=\vec{c}+t\vec{b}$ is
$$d = \frac{|(\vec{c}-\vec{a})\times\vec{b}|}{|\vec{b}|}, \text{ where } |\vec{b}| \neq 0.$$
- If two lines $\frac{x-x_1}{b_1}=\frac{y-y_1}{b_2}=\frac{z-z_1}{b_3}$ and $\frac{x-x_2}{d_1}=\frac{y-y_2}{d_2}=\frac{z-z_2}{d_3}$ intersect, then
$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0.$$
- A straight line which is perpendicular to a plane is called a normal to the plane.
- The equation of the plane at a distance p from the origin and perpendicular to the unit normal vector $\hat{d}$ is $\vec{r}.\hat{d}=p$ (normal form)
- Cartesian equation of the plane in normal form is $\ell x+my+nz=p$.
- Vector form of the equation of a plane passing through a point with position vector $\vec{a}$ and perpendicular to $\vec{n}$ is $(\vec{r}-\vec{a}).\vec{n}=0$
- Cartesian equation of a plane normal to a vector with direction ratios a,b,c and passing through a given point (x_1,y_1,z_1) is $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$
- Intercept form of the equation of the plane $\vec{r}.\vec{n}=q$, having intercepts a, b, c on the x, y, z axes respectively is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.
- Parametric form of vector equation of the plane passing through three given non-collinear points is $\vec{r}=\vec{a}+s(\vec{b}-\vec{a})+t(\vec{c}-\vec{a})$
- Cartesian equation of the plane passing through three non-collinear points is
$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0.$$
- A straight line will lie on a plane if every point on the line, lie in the plane and the normal to the plane is perpendicular to the line.
- The two given non-parallel lines $\vec{r}=\vec{a}+s\vec{b}$ and $\vec{r}=\vec{c}+t\vec{d}$ are coplanar if $(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})=0$
- Two lines $\frac{x-x_1}{b_1}=\frac{y-y_1}{b_2}=\frac{z-z_1}{b_3}$ and $\frac{x-x_2}{d_1}=\frac{y-y_2}{d_2}=\frac{z-z_2}{d_3}$ are coplanar if
$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$
- Non-parametric form of vector equation of the plane containing the two coplanar lines $\vec{r}=\vec{a}+s\vec{b}$ and $\vec{r}=\vec{c}+t\vec{d}$ is $(\vec{r}-\vec{a}).(\vec{b}\times\vec{d})=0$ or $(\vec{r}-\vec{c}).(\vec{b}\times\vec{d})=0$.

- The acute angle θ between the two planes $\vec{r}.\vec{n}_1 = p_1$ and $\vec{r}.\vec{n}_2 = p_2$ is $\theta = \cos^{-1}\left(\frac{|\vec{n}_1.\vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}\right)$
- If θ is the acute angle between the line $\vec{r} = \vec{a} + t\vec{b}$ and the plane $\vec{r}.\vec{n} = p$, then $\theta = \sin^{-1}\left(\frac{|\vec{b}.\vec{n}|}{|\vec{b}||\vec{n}|}\right)$
- The perpendicular distance from a point with position vector $\vec{u}$ to the plane $\vec{r}.\vec{n} = p$ is given by $\delta = \frac{|\vec{u}.\vec{n} - p|}{|\vec{n}|}$
- The perpendicular distance from a point (x_1, y_1, z_1) to the plane $ax + by + cz = p$ is $\delta = \frac{|ax_1 + by_1 + cz_1 - p|}{\sqrt{a^2 + b^2 + c^2}}$.
- The perpendicular distance from the origin to the plane $ax + by + cz + d = 0$ is given by $\delta = \frac{|d|}{\sqrt{a^2 + b^2 + c^2}}$
- The distance between two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is given by $\frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$
- The vector equation of a plane which passes through the line of intersection of the planes $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ is given by $(\vec{r}.\vec{n}_1 - d_1) + \lambda(\vec{r}.\vec{n}_2 - d_2) = 0$, Where $\lambda \in R$.
- The equation of a plane passing through the line of intersection of the planes $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$ is given by $(a_1x + b_1y + c_1z - d_1) + \lambda(a_2x + b_2y + c_2z - d_2) = 0$.
- The position vector of the point of intersection of the line $\vec{r} = \vec{a} + t\vec{b}$ and the plane $\vec{r}.\vec{n} = p$ is $\vec{u} = \vec{a} + \left(\frac{p - (\vec{a}.\vec{n})}{\vec{b}.\vec{n}}\right)\vec{b}$, where $\vec{b}.\vec{n} \neq \vec{0}$
- If $\vec{v}$ is the position vector of the image of $\vec{u}$ in the plane $\vec{r}.\vec{n} = p$, then $\vec{v} = \vec{u} + \frac{2[p - (\vec{u}.\vec{n})]}{|\vec{n}|^2}\vec{n}$

BOOK BACK ONE MARKS

1. If $\vec{a}$ and $\vec{b}$ are parallel vectors, then $[\vec{a}, \vec{c}, \vec{b}]$ is equal to

(1) 2 (2) –1
(3) 1 **(4) 0**

Solution :

$\vec{a} \parallel \vec{b} \Rightarrow \vec{a} = \lambda \vec{b}$

$[\vec{a}\vec{b}\vec{c}] = [\lambda\vec{b}\ \vec{b}\ \vec{c}]$

$= \lambda[\vec{b}\ \vec{b}\ \vec{c}]$

$= 0$

2. If a vector $\vec{\alpha}$ lies in the plane of $\vec{\beta}$ and $\vec{\gamma}$, then

(1) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = 1$ (2) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = -1$
(3) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = 0$ (4) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = 2$

Solution :

Since $\vec{\alpha}$ lies in the plane of $\vec{\beta}$ and $\vec{\gamma}$

$\vec{\alpha} \cdot [\vec{\beta} \times \vec{\gamma}] = 0$

$[\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}] = 0$

3. If $\vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a} = 0$ then the value of $[\vec{a}, \vec{b}, \vec{c}]$ is

(1) $|\vec{a}||\vec{b}||\vec{c}|$ (2) $\frac{1}{3}|\vec{a}||\vec{b}||\vec{c}|$

(3) 1 (4) –1

Solution :

$\vec{a}.\vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}$

$\vec{b}.\vec{c} = 0 \Rightarrow \vec{b} \perp \vec{c}$

$\vec{a}.\vec{c} = 0 \Rightarrow \vec{a} \perp \vec{c}$

$\vec{a}, \vec{b}, \vec{c}$ are mutually $\perp$ vectors

$[\vec{a}, \vec{b}, \vec{c}] = |\vec{a}|\ |\vec{b}|\ |\vec{c}|$

4. If $\vec{a}, \vec{b}, \vec{c}$ are three unit vectors such that $\vec{a}$ is perpendicular to $\vec{b}$, and is parallel to $\vec{c}$ then $\vec{a} \times (\vec{b} \times \vec{c})$ is equal to

(1) $\vec{a}$ **(2) $\vec{b}$**

(3) $\vec{c}$ (4) $\vec{0}$

Solution :

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

$= (\vec{a}.\vec{c})\vec{b} - 0$

$= \vec{b}$

Hint : $\vec{a} \perp \vec{b} \Rightarrow \vec{a}.\vec{b} = 0$

$\vec{a} \parallel \vec{c}$

$\vec{a}.\vec{c} = |\vec{a}||\vec{c}| \cos\theta$

$\vec{a}.\vec{c} = 1$

5. If $[\vec{a}, \vec{b}, \vec{c}] = 1$ then the value of $\frac{\vec{a}.(\vec{b} \times \vec{c})}{(\vec{c} \times \vec{a}).\vec{b}} + \frac{\vec{b}.(\vec{c} \times \vec{a})}{(\vec{a} \times \vec{b}).\vec{c}} + \frac{\vec{c}.(\vec{a} \times \vec{b})}{(\vec{c} \times \vec{b}).\vec{a}}$ is

(1) 1 (2) –1
(3) 2 (4) 3

Solution :

$\frac{\vec{a}.(\vec{b} \times \vec{c})}{(\vec{c} \times \vec{a}).\vec{b}} + \frac{\vec{b}.(\vec{c} \times \vec{a})}{(\vec{a} \times \vec{b}).\vec{c}} + \frac{\vec{c}.(\vec{a} \times \vec{b})}{(\vec{c} \times \vec{b}).\vec{a}}$

$= \frac{[\vec{a}\vec{b}\vec{c}]}{[\vec{a}\vec{b}\vec{c}]} + \frac{[\vec{a}\vec{b}\vec{c}]}{[\vec{a}\vec{b}\vec{c}]} + \frac{[\vec{a}\vec{b}\vec{c}]}{-[\vec{a}\vec{b}\vec{c}]}$

$= 1 + 1 - 1$

$= 1$

Book Answer is wrong

6. The volume of the parallelepiped with its edges represented by the vectors $\hat{i}+\hat{j}, \hat{i}+2\hat{j}, \hat{i}+\hat{j}+\pi\hat{k}$ is

(1) $\frac{\pi}{2}$ (2) $\frac{\pi}{3}$

(3) π (4) $\frac{\pi}{4}$

Solution :

$$\text{volume} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & \pi \end{vmatrix}$$

$$= 1\,(2\pi - 0) - 1\,(\pi - 0) + 0(1-2)$$

$$= 2\pi - \pi + 0 = \pi$$

7. If $\vec{a}$ and $\vec{b}$ are unit vectors such that $[\vec{a}, \vec{b}, \vec{a} \times \vec{b}] = \frac{\pi}{4}$, then the angle between $\vec{a}$ and $\vec{b}$ is

(1) $\frac{\pi}{6}$ (2) $\frac{\pi}{4}$

(3) $\frac{\pi}{3}$ (4) $\frac{\pi}{2}$

Solution :

$$[\vec{a}, \vec{b}, \vec{a} \times \vec{b}] = \frac{\pi}{4}$$

$$\vec{a}.[\vec{b} \times (\vec{a} \times \vec{b})] = \frac{\pi}{4}$$

$$\vec{a}.[(\vec{b}.\vec{b})\vec{a} - (\vec{b}.\vec{a})\vec{b}] = \frac{\pi}{4}$$

$$\vec{a}.(\vec{a} - \vec{b}(\vec{a}.\vec{b})) = \frac{\pi}{4}$$

$$\vec{a}.\vec{a} - (\vec{a}.\vec{b})\,(\vec{a}.\vec{b}) = \frac{\pi}{4}$$

$$1 - (|\vec{a}||\vec{b}|\cos\theta)^2 = \frac{\pi}{4}$$

$$1 - (1.1\cos\theta)^2 = \frac{\pi}{4}$$

$$1 - \cos^2\theta = \frac{\pi}{4}$$

$$\sin^2\theta = \frac{\pi}{4}$$

$$\sin^2\theta = \frac{1}{4} \text{ (Replacing } \frac{\pi}{4} \text{ by } \frac{1}{4}\text{)}$$

$$\sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}$$

8. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + \hat{j}$, $\vec{c} = \hat{i}$ and $(\vec{a} \times \vec{b}) \times \vec{c} = \lambda\vec{a} + \mu\vec{b}$, then the value of $\lambda + \mu$ is

(1) 0 (2) 1

(3) 6 (4) 3

Solution :

$$(\vec{a} \times \vec{b}) \times \vec{c} = \lambda\vec{a} + \mu\vec{b}$$

$$(\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a} = \lambda\vec{a} + \mu\vec{b}$$

$$\vec{b} - \vec{a} = \lambda\vec{a} + \mu\vec{b}$$

$$-\vec{a} + \vec{b} = \lambda\vec{a} + \mu\vec{b}$$

$$\lambda = -1 \qquad \mu = 1$$

$$\lambda + \mu = -1 + 1 = 0$$

9. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar, non-zero vectors such that $[\vec{a}, \vec{b}, \vec{c}] = 3$, then $\{[\vec{a} \times \vec{b}, \vec{b} \times \vec{c}, \vec{c} \times \vec{a}]\}^2$ is equal to

(1) 81 (2) 9

(3) 27 (4) 18

Solution :

$$\{[\vec{a} \times \vec{b}\ \ \vec{b} \times \vec{c}\ \ \vec{c} \times \vec{a}]\}^2 = [[\vec{a}\,\vec{b}\,\vec{c}]^2]^2$$

$$= (3^2)^2$$

$$= (9)^2$$

$$= 81$$

10. If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors such that $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\vec{b}+\vec{c}}{\sqrt{2}}$, then the angle between $\vec{a}$ and $\vec{b}$ is

(1) $\frac{\pi}{2}$ (2) $\frac{3\pi}{4}$

(3) $\frac{\pi}{4}$ (4) π

Solution :

$\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\vec{b}+\vec{c}}{\sqrt{2}}$

$(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} = \frac{1}{\sqrt{2}}(\vec{b}) + \frac{1}{\sqrt{2}}(\vec{c})$

$-(\vec{a}.\vec{b}) = \frac{1}{\sqrt{2}}$

$\vec{a}.\vec{b} = -\frac{1}{\sqrt{2}}$

$\cos\theta = \cos\left(\pi - \frac{\pi}{4}\right)$

$\theta = \frac{3\pi}{4}$ (θ is obtuse)

11. If the volume of the parallelopiped with $\vec{a} \times \vec{b}, \vec{b} \times \vec{c}, \vec{c} \times \vec{a}$ as coterminous edges is 8 cubic units, then the volume of the parallelopiped with $(\vec{a} \times \vec{b}) \times (\vec{b} \times \vec{c})$, $(\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a})$ and $(\vec{c} \times \vec{a}) \times (\vec{a} \times \vec{b})$ as coterminous edges is,

(1) 8 cubic units (2) 512 cubic units

(3) 64 cubic units (4) 24 cubic units

Solution :

$[(\vec{a} \times \vec{b}) \times (\vec{b} \times \vec{c}), (\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a}), (\vec{c} \times \vec{a}) \times (\vec{a} \times \vec{b})]$

$= [\vec{a} \times \vec{b} \;\; \vec{b} \times \vec{c} \;\; \vec{c} \times \vec{a}]^2$

$= (8)^2$

$= 64$

12. Consider the vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ such that $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{0}$. Let P_1 and P_2 be the planes determined by the pairs of vectors $\vec{a}, \vec{b}$ and $\vec{c}, \vec{d}$ respectively. Then the angle between P_1 and P_2 is

(1) 0° (2) 45°

(3) 60° (4) 90°

Solution :

$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{0}$

$(\vec{a} \times \vec{b}) \parallel (\vec{c} \times \vec{d})$

$\theta = 0°$

13. If $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$, where $\vec{a}, \vec{b}, \vec{c}$ are any three vectors such that $\vec{b}.\vec{c} \neq 0$ and $\vec{a}.\vec{b} \neq 0$, then $\vec{a}$ and $\vec{c}$ are

(1) perpendicular

(2) parallel

(3) inclined at an angle $\frac{\pi}{3}$

(4) inclined at an angle $\frac{\pi}{6}$

Solution :

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$

$(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

$(\vec{a}.\vec{b})\vec{c} = (\vec{b}.\vec{c})\vec{a}$

$\vec{c} = \left(\frac{\vec{b}.\vec{c}}{\vec{a}.\vec{b}}\right)\vec{a}$

$\vec{c} = \lambda \vec{a}$

$\vec{c} \parallel \vec{a}$

14. If $\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-5\hat{k}$, $\vec{c}=3\hat{i}+5\hat{j}-\hat{k}$, then a vector perpendicular to $\vec{a}$ and lies in the plane containing $\vec{b}$ and $\vec{c}$ is

(1) $-17\hat{i}+21\hat{j}-97\hat{k}$

(2) $17\hat{i}+21\hat{j}-123\hat{k}$

(3) $-17\hat{i}-21\hat{j}+97\hat{k}$

(4) $-17\hat{i}-21\hat{j}-97\hat{k}$

Solution :

$$\vec{b}\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & -5\\ 3 & 5 & -1\end{vmatrix}$$

$$=\hat{i}(-2+25)-\hat{j}(-1+15)+\hat{k}(5-6)$$

$$\vec{b}\times\vec{c}=23\hat{i}-14\hat{j}-\hat{k}$$

$$\vec{a}\times(\vec{b}\times\vec{c})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & -1\\ 23 & -14 & -1\end{vmatrix}$$

$$=\hat{i}(-3-14)-\hat{j}(-2+23)+\hat{k}(-28-69)$$

$$=-17\hat{i}-21\hat{j}-97\hat{k}$$

15. The angle between the lines $\frac{x-2}{3}=\frac{y+1}{-2}, z=2$ and $\frac{x-1}{1}=\frac{2y+3}{3}=\frac{z+5}{2}$ is

(1) $\frac{\pi}{6}$ (2) $\frac{\pi}{4}$

(3) $\frac{\pi}{3}$ (4) $\frac{\pi}{2}$

Solution :

$$\vec{b}=3\hat{i}-2\hat{j}$$

$$\vec{c}=\hat{i}+\frac{3}{2}\hat{j}+2\hat{k}$$

$$\vec{b}.\vec{c}=3-2\left(\frac{3}{2}\right)+0$$

$$\vec{b}.\vec{c}=0 \Rightarrow \vec{b}\perp\vec{c}$$

$$\theta=90^\circ \quad (\text{or}) \ \frac{\pi}{2}$$

16. If the line $\frac{x-2}{3}=\frac{y-1}{-5}=\frac{z+2}{2}$ lies in the plane $x+3y-\alpha z+\beta=0$, then (α,β) is

(1) $(-5,5)$ (2) $(-6,7)$

(3) $(5,-5)$ (4) $(6,-7)$

Solution :

$$\vec{b}=3\hat{i}-5\hat{j}+2\hat{k}$$

$$\vec{n}=\hat{i}+3\hat{j}-\alpha\hat{k}$$

$$\hat{b}.\hat{n}=0$$

$$3-15-2\alpha=0$$

$$-2\alpha=12$$

$$\alpha=-6$$

Also $(2, 1, -2)$ lies on plane $x+3y-\alpha z+\beta=0$

$$2+3-12+\beta=0$$

$$-7+\beta=0$$

$$\beta=7$$

$$(\alpha,\beta)=(-6,7)$$

17. The angle between the line $\vec{r}=(\hat{i}+2\hat{j}-3\hat{k})+t(2\hat{i}+\hat{j}-2\hat{k})$ and the plane $\vec{r}.(\hat{i}+\hat{j})+4=0$ is

(1) 0° (2) 30°

(3) 45° (4) 90°

Solution :

$$\vec{b}=2\hat{i}+\hat{j}-2\hat{k}$$

$$\vec{n}=\hat{i}+\hat{j}$$

$$|\vec{b}|=\sqrt{4+1+4}=\sqrt{9}=3$$

$$|\vec{n}|=\sqrt{1+1}$$

$$|\vec{n}|=\sqrt{2}$$

$$\vec{b}.\vec{n}=2+1=3$$

$$\sin\theta = \left(\frac{|\vec{b}.\vec{n}|}{|\vec{b}||\vec{n}|}\right) = \frac{3}{3\sqrt{2}}$$

$$\sin\theta = \frac{1}{\sqrt{2}}$$

$$\theta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$\theta = 45°$$

18. The coordinates of the point where the line $\vec{r} = (6\hat{i} - \hat{j} - 3\hat{k}) + t(-\hat{i} + 4\hat{k})$ meets the plane $\vec{r}.(\hat{i} + \hat{j} - \hat{k}) = 3$ are

(1) (2,1,0) (2) (7,–1,–7)

(3) (1,2,–6) **(4) (5,–1,1)**

Solution :

$$\frac{x-6}{-1} = \frac{y+1}{0} = \frac{z+3}{4} = \lambda$$

Point of this form $(-\lambda + 6, -1, 4\lambda - 3)$

This lies on the plane $x + y - z - 3 = 0$

$$-\lambda + 6 - 1 - 4\lambda + 3 - 3 = 0$$

$$-5\lambda = -5$$

$$\lambda = 1$$

Point of intersection (5, –1, 1)

19. Distance from the origin to the plane $3x - 6y + 2z + 7 = 0$ is

(1) 0 **(2) 1**

(3) 2 (4) 3

Solution :

$$\text{Distance} = \frac{7}{\sqrt{(3)^2 + (-6)^2 + (2)^2}}$$

$$= \frac{7}{\sqrt{9+36+4}} = \frac{7}{7} = 1$$

20. The distance between the planes $x + 2y + 3z + 7 = 0$ and $2x + 4y + 6z + 7 = 0$ is

(1) $\frac{\sqrt{7}}{2\sqrt{2}}$ (2) $\frac{7}{2}$

(3) $\frac{\sqrt{7}}{2}$ (4) $\frac{7}{2\sqrt{2}}$

Solution :

The planes are

$$x + 2y + 3z + 7 = 0$$

$$x + 2y + 3z + \frac{7}{2} = 0$$

$$\text{Distance} = \frac{\left|7 - \frac{7}{2}\right|}{\sqrt{(1)^2 + (2)^2 + (3)^2}} = \frac{\frac{7}{2}}{\sqrt{1+4+9}}$$

$$= \frac{7}{2\sqrt{14}} = \frac{\sqrt{7}\sqrt{7}}{2\sqrt{2}\sqrt{7}}$$

$$= \frac{\sqrt{7}}{2\sqrt{2}}$$

21. If the direction cosines of a line are $\frac{1}{c}, \frac{1}{c}, \frac{1}{c}$ then

(1) $c = \pm 3$ **(2) $c = \pm\sqrt{3}$**

(3) $c > 0$ (4) $0 < c < 1$

Solution :

DC'S of a line $\frac{1}{c}, \frac{1}{c}, \frac{1}{c}$ then

$$(\cos\alpha, \cos\beta, \cos\gamma) = \left(\frac{1}{c}, \frac{1}{c}, \frac{1}{c}\right)$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\frac{1}{c^2} + \frac{1}{c^2} + \frac{1}{c^2} = 1$$

$$\frac{3}{c^2} = 1$$

$$c^2 = 3$$

$$c = \pm\sqrt{3}$$

22. The vector equation $\vec{r} = (\hat{i} - 2\hat{j} - \hat{k}) + t(6\hat{j} - \hat{k})$ represents a straight line passing through the points

(1) (0, 6, –1) and (1, –2, –1)

(2) (0, 6, –1) and (–1, –4, –2)

(3) (1, –2, –1) and (1, 4, –2)

(4) (1, –2, –1) and (0, –6, 1)

Solution :

$\vec{a} = \hat{i} - 2\hat{j} - \hat{k}$

Cartesian form

$$\frac{x-1}{0} = \frac{y+2}{6} = \frac{z+1}{-1} = \lambda$$

Any point of this form (1, 6λ–2, –λ–1)

$\lambda = 0$ then (1, –2, –1)

$\lambda = 1$ then (1, 4, –2)

23. If the distance of the point (1,1,1) from the origin is half of its distance from the plane $x + y + z + k = 0$, then the values of k are

(1) ±3 (2) ±6

(3) –3, 9 **(4) 3, –9**

Solution :

Distance from (1, 1, 1) to (0, 0, 0)

$$= \frac{1}{2}\left|\frac{x_1 + y_1 + z_1 + k}{\sqrt{(1)^2 + (1)^2 + (1)^2}}\right|$$

$$\sqrt{(1)^2 + (1)^2 + (1)^2} = \frac{1}{2}\left|\frac{3+k}{\sqrt{3}}\right|$$

$$2\sqrt{3}\sqrt{3} = |3+k|$$

$$|3+k| = 6$$

$$3 + k = \pm 6$$

$k = -3 + 6$ $k = -3 - 6$

$k = 3$ $k = -9$

24. If the planes $\vec{r}.(2\hat{i} - \lambda\hat{j} + \hat{k}) = 3$ and $\vec{r}.(4\hat{i} + \hat{j} - \mu\hat{k}) = 5$ are parallel, then the value of λ and μ are

(1) $\frac{1}{2}, -2$ (2) $\frac{-1}{2}, 2$

(3) $-\frac{1}{2}, -2$ (4) $\frac{1}{2}, 2$

Solution :

S $\vec{n_1} = \vec{n_2}$

S $(2\hat{i} - \lambda\hat{j} + \hat{k}) = 4\hat{i} + \hat{j} - \mu\hat{k}$

S $= 2$

$4\hat{i} - 2\lambda\hat{j} + 2\hat{k} = 4\hat{i} + \hat{j} - \mu\hat{k}$

$-2\lambda = 1$ $-\mu = 2$

$\lambda = -\frac{1}{2}$ $\mu = -2$

25. If the length of the perpendicular from the origin to the plane $2x + 3y + \lambda z = 1$, $\lambda > 0$ is $\frac{1}{5}$, then the value of λ is

(1) $2\sqrt{3}$ (2) $3\sqrt{2}$

(3) 0 (4) 1

Solution :

Distance $= \frac{1}{5}$

$$\left|\frac{2x_1 + 3y_1 + \lambda z_1 - 1}{\sqrt{2^2 + 3^2 + \lambda^2}}\right| = \frac{1}{5}$$

$$\left|\frac{-1}{\sqrt{4 + 9 + \lambda^2}}\right| = \frac{1}{5}$$

$$5^2 = 13 + \lambda^2$$

$$\lambda^2 = 25 - 13$$

$$\lambda^2 = 12$$

$$\lambda = \pm\sqrt{12}$$

$$\lambda = \pm 2\sqrt{3}$$